

# The resource theory of quantum reference frames: manipulations and monotones

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Every restriction on quantum operations defines a resource theory, determining how quantum states that cannot be prepared under the restriction may be manipulated and used to circumvent the restriction. A superselection rule is a restriction that arises through the lack of a classical reference frame and the states that circumvent it (the resource) are quantum reference frames. We consider the resource theories that arise from three types of superselection rule, associated respectively with lacking: (i) a phase reference, (ii) a frame for chirality, and (iii) a frame for spatial orientation. Focussing on pure unipartite quantum states (and in some cases restricting our attention even further to subsets of these), we explore single-copy and asymptotic manipulations. In particular, we identify the necessary and sufficient conditions for a deterministic transformation between two resource states to be possible and, when these conditions are not met, the maximum probability with which the transformation can be achieved. We also determine when a particular transformation can be achieved reversibly in the limit of arbitrarily many copies and find the maximum rate of conversion. A comparison of the three resource theories demonstrates that the extent to which resources can be interconverted decreases as the strength of the restriction increases. Along the way, we introduce several measures of framedness and prove that these are monotonically nonincreasing under various classes of operations that are permitted by the superselection rule.

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## I. INTRODUCTION

For every interesting restriction on operations, there is a resulting resource theory [1]. For instance, the restriction of local operations and classical communication (LOCC) leads to the theory of entanglement. Against the backdrop of the LOCC restriction (and an implicit restriction that the parties do not share any entanglement at the outset), a single entangled pair or a single use of a noiseless quantum channel are both considered resources, one static, the other dynamic [2]. A resource theory specifies the manner in which one can inter-convert between various resources, for instance, whether one entangled state can be transformed by LOCC into another. Indeed, much of quantum information theory is simply a theory of the inter-conversion between resources [3]. We are here interested in the restriction of a superselection rule (SSR). Specifically, we imagine a party that is restricted to operations that are invariant under the action of a group  $G$  and refer to this as a *superselection rule for  $G$*  or simply a  $G$ -SSR. Although SSRs are often considered to be axiomatic, it is better to consider them as arising from practical restrictions. Indeed, most SSRs can be lifted if one has access to a reference frame for the group in question, so the restriction is ultimately one of access to an appropriate reference frame [4, 5, 6]. In this context, the analogue of an entangled state – that which can be used to temporarily overcome the LOCC restriction – is a state that can be used to temporarily overcome the restriction of the SSR. Such states have been referred to as *bounded-size reference frames* or simply *quantum reference frames*. In this article, we study the manner in which one can interconvert between such states under the SSR. We are therefore exploring the resource theory of quantum reference frames.

Just as we say that a state is entangled or has entanglement if it cannot be prepared by LOCC, one can say that a state is  $G$ -asymmetric or has nonzero  $G$ -frameness if it cannot be prepared by  $G$ -invariant operations. One of the goals of this article is to provide operational measures of frameness. The minimal requirement on such a measure is that it be monotonically nonincreasing under  $G$ -invariant operations, in which case it will be called a  $G$ -frameness monotone, in analogy with the requirement that entanglement measures be monotonically nonincreasing under LOCC operations. We distinguish three sorts of monotones: deterministic, ensemble, and stochastic. These correspond respectively to monotonicity under deterministic operations between states, under deterministic operations between states and ensembles of states, and under stochastic operations between states. An ensemble monotone is the standard notion of a monotone in entanglement theory (with LOCC standing in for  $G$ -invariant operations), while a deterministic monotone has been recently studied in entanglement theory under the name of a type 2 monotone [7]. The notion of an ensemble frameness monotone is present (though unnamed) in Vaccaro *et al.* [8] and Schuch *et al.* [9, 10] while that

of a deterministic frameness monotone is found in Appendix A of Ref. [11]. We provide examples of each sort of monotone through the various SSRs we consider. As in entanglement theory, a focus on monotonicity properties is motivated by its utility in the study of frame manipulations.

The structure of any quantum resource theory is dependent on the extent of the restriction; the more restricted the set of allowed operations, the fewer possibilities there are of inter-conversions among different forms of a resource. For instance, in entanglement theory, the restriction to LOCC operations is more substantive the more parties one considers. Consider the question of whether two pure entangled states can be interconverted in the sense of being transformed one to the other with some nonzero probability using stochastic LOCC [12]. For bipartite pure states, one finds that any two entangled states are interconvertible in this sense. However, in the tripartite case, one finds that the intrinsically 3-way entangled pure states are divided into two classes, the GHZ and the W states, with interconvertibility being possible only within but not between the classes. For intrinsically 4-way entangled pure states, the number of classes becomes infinite [13, 14]. Thus  $n$ -way LOCC constrains manipulations among  $n$ -partite pure entangled states more strongly than  $(n - 1)$ -way LOCC constrains manipulations among  $(n - 1)$ -partite pure entangled states. The higher the number of parties, the stronger is the constraint of LOCC.

Similarly, as we demonstrate in this article, an increase in the strength of the superselection rule (where one SSR is stronger than another if it allows fewer operations), leads to a decrease in the number of possibilities for interconversion among quantum reference frames. In particular, we show that this is the case as one progresses through the relatively mild restriction of a  $Z_2$ -SSR, to the stronger restriction of a  $U(1)$ -SSR, to the very strong restriction of an  $SU(2)$ -SSR. Each resource theory has its own section in this article. Sections III, IV and V deal respectively with the resource theories for the  $U(1)$ -SSR, the  $Z_2$ -SSR, and the  $SU(2)$ -SSR. (The  $U(1)$  case is considered first because it is likely to be the most familiar and it is the one upon which the most previous work has been done.) There are many ways in which each SSR may arise in practice, and a particular example is provided for each. Specifically, the  $Z_2$ -SSR is shown to correspond to lacking a reference frame for chirality, the  $U(1)$ -SSR to lacking a phase reference, and the  $SU(2)$ -SSR to lacking a Cartesian frame.

We consider various types of manipulations of pure states for each of these resource theories. In the context of single-copy manipulations, we seek to determine necessary and sufficient conditions for a transformation from one pure state to another to be possible by deterministic operations under the SSR. These results play the same role in the resource theory of quantum reference frames as Nielsen's theorem plays in the theory of entanglement [15]. If a transformation between two pure states

is not possible deterministically, we attempt to find the maximum probability with which the conversion can be achieved. Our results on this front provide the analogues for reference frames of Vidal's theorem in entanglement theory [16].

A comparison of these results provides one of the senses in which the stronger SSRs allow fewer frame manipulations. For instance, we can ask, for every type of SSR, whether the stochastic invariant operations define multiple different classes in the sense of stochastic interconvertibility being possible only within but not between the classes.

The case of the  $Z_2$ -SSR is similar to that of pure bipartite entanglement: for every pair of resource states, one member of the pair can be converted to the other with some probability. Actually, the resource theory of the  $Z_2$ -SSR is even nicer than the theory of pure bipartite entanglement. In the latter, the reverse of a stochastically-achievable conversion need not be stochastically-achievable (for instance if the first state has a larger Schmidt number than the second), whereas under the  $Z_2$ -SSR, for every pair of states there is a nonzero probability of converting both the first to the second and the second to the first.

The amount of interconvertibility is reduced in the case of the  $U(1)$ -SSR. For a given pair of resource states, it need not be the case that one member of the pair can be converted to the other. For instance, a single copy of  $|0\rangle + |1\rangle$  cannot be converted to a single copy of  $|0\rangle + |2\rangle$ , or vice-versa, with any probability (where  $|n\rangle$  denotes an eigenstate of the number operator). If we introduce an ordering relation among states wherein one state is judged higher than another if it can be converted to the other with some probability, then the states form a partially ordered set under the  $U(1)$ -SSR. The pair of  $|0\rangle + |1\rangle$  and  $|0\rangle + |2\rangle$  provide an example of two elements that are not ordered. (Note, however, that for every pair of states, there is a third that is above both in the partial order. For instance, both  $|0\rangle + |1\rangle$  and  $|0\rangle + |2\rangle$  can be obtained with some probability from the state  $|0\rangle + |1\rangle + |2\rangle$ .) This is similar to the situation that exists in the theory of pure *tripartite* entanglement, where GHZ states and W states cannot be interconverted one to the other with any probability.

The amount of interconvertibility is reduced even further in the case of the  $SU(2)$ -SSR. Under the  $U(1)$ -SSR, the pair of states  $|2\rangle + |3\rangle$  and  $|0\rangle + |1\rangle + |2\rangle$  are ordered with respect to one another: although the latter can't be obtained from the former with any probability, the opposite conversion is possible. However, an analogous pair of resource states under the  $SU(2)$ -SSR,  $|2, 2\rangle + |3, 3\rangle$  and  $|0, 0\rangle + |1, 1\rangle + |2, 2\rangle$  (where  $|j, m\rangle$  denotes the joint eigenstate of  $J^2$  and  $J_z$  with eigenvalues  $j(j+1)$  and  $\hbar m$ ) are not ordered with respect to one another as neither can be obtained from the other with any probability.

We also consider *asymptotic* manipulations of pure states. The question here is: given an arbitrarily large number of copies of one pure state, with what rate can

one deterministically transform these to (a good approximation of) an arbitrarily large number of copies of a different pure state under the SSR? If the asymptotic interconversion can be achieved *reversibly* between any two states, then a unique measure of framedness over the states is sufficient to characterize the rate of interconversion. In the theory of bipartite entanglement for pure states, the entropy of entanglement is such a measure. We demonstrate that a unique measure also exists in the resource theory for the  $Z_2$ -SSR. Under the  $U(1)$ -SSR, we find that certain types of states cannot be asymptotically interconverted one to the other at any rate (for instance, one cannot distill  $|0\rangle + |1\rangle$  from  $|0\rangle + |2\rangle$ ). However, we show that for a large class of states, reversible interconversion *is* possible, and the unique measure of framedness that determines the rate of interconversion is simply the variance over number (the connection of this result to the one of Ref. [10] is discussed below). In the resource theory of the  $SU(2)$ -SSR, we again find that there are pairs of states for which the rate of distillation of one from the other is strictly zero. In contrast with our  $U(1)$  case, however, we can identify classes of states for which there is a nonzero rate of interconversion in both directions, but where for certain pairs the rate in one direction is not the inverse of the rate in the other. It follows that a single measure of framedness is not in general sufficient to infer the rate of interconversion of one state to another in this class. Nonetheless, we show that a *pair* of measures is sufficient to infer the rates. Furthermore, although there still exist subclasses of states for which reversible interconversion is possible, these are much smaller than those defined by the  $U(1)$ -SSR.

Another feature of the resource theory of quantum reference frames that does not have any analogue in the theory of pure bipartite entanglement is found in the asymptotic manipulation of resources under the  $Z_2$ -SSR. As we have said, any two states can be reversibly interconverted asymptotically under the  $Z_2$ -SSR, in analogy with pure bipartite entanglement theory. However, unlike the latter, the rate of interconversion fails to be an ensemble monotone. This result calls into question the widespread tendency to require ensemble monotonicity of any measure of a resource (such as entanglement or framedness) and is an example of how the study of quantum reference frames may yield insights into which features of entanglement theory are generic to resource theories and which not.

Of course, a prerequisite to answering all the sorts of questions we have just described is a characterization of the full set of generalized operations that are permitted under the SSR, that is, the full set of allowed trace-nonincreasing completely positive maps. Therefore, at the outset of this article, we demonstrate that a  $G$ -invariant operation can be characterized as one that admits a Kraus decomposition in terms of *irreducible tensor operators* (see for example p. 232 in [17]) for the group  $G$ . This connection allows us to provide convenient expressions for the invariant operations. In particular, in

the context of the  $SU(2)$ -SSR, where the characterization is particularly difficult, the Wigner-Eckart theorem (a well-known result in nuclear physics [17]) specifies the form that irreducible tensor operators may take.

We end this introduction by placing this article in the context of previous work in this area. There has been substantial progress on the theory of quantum reference frames in the last few years. A problem that has seen a great deal of attention is that of identifying the optimal state of a quantum reference frame for transmitting information about some degree of freedom (such as chirality, phase, or orientation) according to some figure of merit (such as the fidelity). The pioneering paper in this field is arguably that of Gisin and Popescu [18], who consider the problem of distributing a single direction in space. This problem, together with that of distributing a triad of orthogonal directions, was subsequently studied intensively by various groups (see Refs. [19, 20, 21]). The problem of the distribution of a phase reference has also received a great deal of attention, with roots in the field of phase estimation [22]. In the case of the problem of distributing chirality, see Refs. [23, 24, 25]. A synthesis of much of this work and more references can be found in Ref. [6].

There has also been a great deal of work on the resource theory of *shared* quantum reference frame, that is, bipartite states that substitute for a reference frame that is common to many spatially separated parties. This research has been essentially confined, however, to the case of phase references. For instance, van Enk [26] considers the interconversion of static and dynamic resources (such as qubits, ebits, cobits, and rebits) in the presence of a  $U(1)$ -SSR, while Bartlett *et al.* [27] demonstrate some analogies between the theory of mixed bipartite entangled states and the theory of pure shared phase references, such as the existence of states that are not locally preparable but from which free singlets cannot be distilled (a phenomenon that was also noted in Refs. [9, 10, 26]). The work of Schuch, Verstraete, and Cirac [9, 10], however, has the most significance for the present article. These authors considered resource manipulations under the following pair of restrictions: (i) only LOCC operations can be implemented, (ii) global and local  $U(1)$ -SSRs are in effect. There is a rich interplay between these two restrictions which is explored in detail in their article. By contrast, we consider the restriction of a  $U(1)$ -SSR alone in a unipartite context. This allows us to identify which aspects of the Schuch *et al.* resource theory are due solely to the  $U(1)$ -SSR and which rely on the further restriction of LOCC.

A statement in Ref. [10] suggests that the resource theory for a unipartite scenario with a  $U(1)$ -SSR (the one we consider here) must be trivial: “the SSR imposes that for any operator  $O$ ,  $[O, \hat{N}] = 0$  must hold [...] As the same restriction holds for the admissible density operators, all states can be converted into each other, and no interesting new effects can be found.” However, this negative assessment is only defensible under the presump-

tion that SSRs are axiomatic restrictions. By contrast, a party who lacks a reference frame for some degree of freedom is restricted to operations that are invariant – the same restriction imposed by an axiomatic SSR – but he faces no restriction on the states of systems that he might come to acquire. To be sure, he cannot prepare arbitrary states himself, but he *can* be provided with systems prepared in such states by another party who has access to the reference frame. Such a state would then constitute a resource. It is for this reason that there is a nontrivial resource theory to be developed even in the unipartite scenario<sup>1</sup>.

Finally, the notion of *generalized entanglement* introduced by Barnum, Knill, Ortiz, Somma, and Viola [28] provides a different approach to the quantification of frameness. In the present work, resource theories are characterized by what parties lack, for instance, a quantum channel or a reference frame. Conversely, the approach of Ref. [28] characterizes a resource theory by specifying the set of observables that parties can access. For instance, choosing the distinguished set to be the *local* observables, it is possible to define a criterion for whether a state is entangled or not. Similarly, by choosing the distinguished set to be the observables that commute with the action of the group associated with a given reference frame, one obtains a criterion for whether a state acts as a quantum reference frame or not. Where the approach of Ref. [28] is lacking, however, is in characterizing the *operations* that define the resource theory. Nonetheless, Ref. [28] includes a preliminary exploration of various possibilities for doing so, and the present article provides further clues. It is possible, therefore, that this framework may be developed into something that is sufficiently general to describe both the resource theories of entanglement and quantum reference frames and possibly many others besides.

## A. Summary of main results

We here provide a brief synopsis of the main results of this article. For each of the three types of SSRs, we characterize the operations (i.e. trace-nonincreasing CP maps) that are invariant under the group  $G$  in question. We state the necessary and sufficient conditions for a transformation of one pure resource state to another to be possible by both deterministic and stochastic  $G$ -invariant operations, and in the latter case, we state what we have found regarding the maximum probability

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<sup>1</sup> There is a sense in which a local reference frame is always a shared reference frame with some other party (although this party may sometimes be quite nebulous, such as the fixed stars). Therefore, it is possible to extract some results for the problem in which we are interested from the results of Schuch *et al.* Nonetheless, we opt instead to derive all of our results directly because we believe this to be a more intuitive approach.

with which the transformation can be achieved. Finally, we describe to what extent a pair of resource states can be interconverted asymptotically. The results are numbered as in the text, although the statement of each might differ slightly. Explanations and proofs can be found in the text.

### 1. $Z_2$ -SSR

The Hilbert space decomposition induced by a unitary representation of  $Z_2$  is simply  $\mathcal{H} = \bigoplus_b \mathcal{H}_b$ , where the bit  $b \in \{0, 1\}$  labels the irreducible representations of  $Z_2$ , and the  $\mathcal{H}_b$  are multiplicity spaces. Again, transformations within the multiplicity spaces are clearly  $Z_2$ -invariant and consequently we can confine our attention to  $\mathcal{H}' = \text{span}\{|b\rangle\} \subseteq \mathcal{H}$  for some arbitrary choice of even parity state  $|0\rangle$  and odd parity state  $|1\rangle$  in each multiplicity space.

**Lemma 9.** A  $Z_2$ -invariant operation admits a Kraus decomposition  $\{K_{B,\alpha}\}$ , where  $B \in \{0, 1\}$  and  $\alpha$  is an integer, satisfying

$$K_{B,\alpha} = S_B \tilde{K}_{B,\alpha} \quad (1)$$

where  $\tilde{K}_{B,\alpha} \equiv c_0^{(B,\alpha)} |0\rangle\langle 0| + c_1^{(B,\alpha)} |1\rangle\langle 1|$  changes the relative amplitudes of the parity states, and  $S_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$  and  $S_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$  do nothing or flip the parity respectively. The coefficients satisfy  $\sum_{B,\alpha} |c_b^{(B,\alpha)}|^2 \leq 1$  for all  $b$ , with equality if the operation is trace-preserving.

We consider transformations between two states,  $|\psi\rangle$  and  $|\phi\rangle$ , that are  $Z_2$ -noninvariant. We define

$$\begin{aligned} p_b &\equiv \langle \psi | \Pi_b | \psi \rangle \\ q_b &\equiv \langle \phi | \Pi_b | \phi \rangle \end{aligned}$$

where  $\Pi_b$  is the projector onto  $\mathcal{H}_b$ . (The assumed  $Z_2$ -noninvariance of  $|\psi\rangle$  and  $|\phi\rangle$  imply that all these weights must be nonzero.) We also define

$$\mathcal{C}(|\psi\rangle) \equiv 2\min\{p_0, p_1\}. \quad (2)$$

The results we have derived are as follows.

**Theorem 10.** The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by a deterministic  $Z_2$ -invariant operation if and only if

$$\mathcal{C}(|\psi\rangle) \geq \mathcal{C}(|\phi\rangle). \quad (3)$$

It turns out that *any* transformation  $|\psi\rangle \rightarrow |\phi\rangle$  can always be achieved with some probability by stochastic  $Z_2$ -invariant operations, so we need only specify the maximum achievable probability.

**Theorem 14.** If  $|\psi\rangle \rightarrow |\phi\rangle$  is not possible by deterministic  $Z_2$ -invariant operations, then the maximum probability of transforming  $|\psi\rangle$  into  $|\phi\rangle$  using  $Z_2$ -invariant operations is

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \frac{\mathcal{C}(|\psi\rangle)}{\mathcal{C}(|\phi\rangle)}.$$

If a set of resource states is such that for every pair, a reversible interconversion of arbitrarily many copies of one to arbitrarily many copies of the other is possible asymptotically with arbitrarily high fidelity, then the maximum rate of any interconversion is fixed by a unique measure (modulo normalization) over the set.

**Theorem 15.** Under the  $Z_2$ -SSR, asymptotic reversible interconversion is possible between any two pure resource states, and the unique asymptotic measure of  $Z_2$ -frameness (modulo normalization) is

$$F^\infty(|\psi\rangle) = -\log |p_0 - p_1|. \quad (4)$$

### 2. $U(1)$ -SSR

A unitary representation of  $U(1)$  induces a decomposition of the Hilbert space  $\mathcal{H}$  of the form  $\mathcal{H} = \bigoplus_n (\mathbb{C} \otimes \mathcal{H}_n)$  where  $n \in \mathbb{N}$  labels the irreducible representations of  $U(1)$  and the  $\mathcal{H}_n$  are multiplicity spaces. Because any change to the multiplicity index does not require a phase reference, we can, without loss of generality, confine our attention to  $\mathcal{H}' = \text{span}\{|n\rangle\} \subseteq \mathcal{H}$  for some arbitrary choice of number state  $|n\rangle$  in each multiplicity space.

**Lemma 2.** An arbitrary  $U(1)$ -invariant operation on  $\mathcal{B}(\mathcal{H}')$  admits a Kraus decomposition  $\{K_{k,\alpha}\}$ , where  $k$  and  $\alpha$  are integers, such that

$$K_{k,\alpha} = S_k \tilde{K}_{k,\alpha}$$

where  $\tilde{K}_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle\langle n|$  changes the relative amplitudes of the different number states, possibly eliminating some, and  $S_k = \sum_{n=\max\{0, -k\}} |n+k\rangle\langle n|$  shifts the number of excitations upward by  $k$ , that is, upward by  $|k|$  if  $k > 0$ , and downward by  $|k|$  if  $k < 0$ . The coefficients satisfy  $\sum_{k,\alpha} |c_n^{(k,\alpha)}|^2 \leq 1$  for all  $n$ , with equality if the operation is trace-preserving.

We now consider transformations between a source state  $|\psi\rangle$  and a target state  $|\phi\rangle$ . Note that these are assumed to be resources, that is, states that are  $U(1)$ -noninvariant. We denote the weights on  $n$  for each of these by

$$\begin{aligned} p_n &\equiv \langle \psi | \Pi_n | \psi \rangle \\ q_n &\equiv \langle \phi | \Pi_n | \phi \rangle \end{aligned}$$

where  $\Pi_n$  is the projector onto  $\mathcal{H}_n$ . We also define the number spectrum of a state  $|\psi\rangle$  by

$$\text{Spec}(\psi) \equiv \{n | p_n \neq 0\},$$

the set of  $n$  that have nonzero weight in  $|\psi\rangle$ .

We derive the following results.

**Theorem 3.** The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by a deterministic  $U(1)$ -invariant operation if and only if  $p_n$  can be obtained from  $q_n$  by a convex sum of shift operations, that is,

$$p_n = \sum_{k=-\infty}^{\infty} w_k q_{n-k},$$

where  $0 \leq w_k \leq 1$  and  $\sum_k w_k = 1$ .

**Theorem 4.** The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by a stochastic U(1)-invariant operation if and only if the number spectrum of  $\phi$  is a subset of the shifted number spectrum of  $\psi$ , that is,

$$\exists k \in \mathbb{Z} : \text{Spec}(\phi) \subset \text{Spec}(\psi) + k$$

where  $\text{Spec}(\psi) + k \equiv \{n + k | p_n \neq 0\}$ ,

**Theorem 5.** If there is only a single value of  $k$  such that the condition  $\text{Spec}(\phi) \subset \text{Spec}(\psi) + k$  holds, then the maximum probability of achieving the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  using U(1)-invariant operations is

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \min_n \left( \frac{p_n}{q_{n+k}} \right).$$

Note that in general there will be multiple values of  $k$  such that  $\text{Spec}(\phi) \subset \text{Spec}(\psi) + k$ . We have not identified the maximum probability in these cases.

To state our results for asymptotic transformations, we must first provide a definition: a number spectrum is said to be *gapless* if the increment between every successive pair of numbers in the spectrum is 1.

**Theorem 7.** Under the U(1)-SSR, asymptotic reversible interconversion is possible among the pure resource states that have a gapless number spectrum. Within this set of states, the unique asymptotic measure of U(1)-freeness is the scaled number variance,

$$F^\infty(|\psi\rangle) \equiv 4 \left[ \langle \psi | \hat{N}^2 | \psi \rangle - \langle \psi | \hat{N} | \psi \rangle^2 \right],$$

and the scaling factor of 4 is chosen so that  $(|0\rangle + |1\rangle)/\sqrt{2}$  has measure 1.

### 3. SU(2)-SSR

The Hilbert space decomposition induced by a unitary representation of SU(2) is  $\mathcal{H} = \bigoplus_j \mathcal{M}_j \otimes \mathcal{N}_j$  where  $j \in \{0, 1/2, 1, 3/2, \dots\}$  labels the irreducible representations of SU(2), the  $\mathcal{M}_j$  are the representation spaces and the  $\mathcal{N}_j$  are the multiplicity spaces. Again, transformations within the multiplicity spaces are SU(2)-invariant and consequently we can confine our attention to  $\mathcal{H}' = \bigoplus_j \mathcal{M}_j = \text{span}\{|j, m\rangle\}_{j,m} \subseteq \mathcal{H}$ , defined by an arbitrary choice of state in each multiplicity space.

**Lemma 17.** An arbitrary SU(2)-invariant operation on  $\mathcal{B}(\mathcal{H}')$  admits a Kraus decomposition  $\{K_{J,M,\alpha}\}$ , where  $J \in \{0, 1/2, 1, 3/2, \dots\}$ ,  $M \in \{-J, \dots, J\}$  and  $\alpha$  is an integer, such that

$$\begin{aligned} K_{J,M,\alpha} = & \sum_{j'=0,1/2,1,\dots} \sum_{m=-j'}^{j'} \sum_{j=|J-j'|}^{J+j'} (-1)^{j'-m} \\ & \times \begin{pmatrix} j' & J & j \\ -m & M & m-M \end{pmatrix} \\ & \times f_{J,\alpha}(j', j) |j', m\rangle \langle j, m-M|. \end{aligned} \quad (5)$$

where the  $2 \times 3$  matrix is a Wigner  $3j$  symbol and the function  $f_{J,\alpha}(j', j)$  does not depend on  $m$  or  $M$ , and satisfies  $\sum_{j,j',\alpha} |f_{J,\alpha}(j', j)|^2 \leq 2j+1$  for all  $j$ , with equality if the operation is trace-preserving.

Rather than solving the resource theory for the SU(2)-SSR in complete generality, we have restricted our attention to a subset of all possible pure states, namely those confined to a subspace  $\mathcal{H}_{\hat{n}} \equiv \text{span}\{|j, j\rangle_{\hat{n}} | j = 0, 1/2, 1, \dots\} \subset \mathcal{H}'$  (where  $|j, j\rangle_{\hat{n}}$  is the highest weight eigenstate of  $\vec{J} \cdot \hat{n}$ ) for some  $\hat{n}$ .  $\mathcal{H}_{\hat{n}}$  is the space of linear combinations of SU(2)-coherent states associated with the quantization axis  $\hat{n}$ . We demonstrate that SU(2)-invariant maps cannot transform a pure state inside  $\mathcal{H}_{\hat{n}}$  to one outside  $\mathcal{H}_{\hat{n}}$  with any probability, and so the only nontrivial resource theory for such states corresponds to transformations within a given  $\mathcal{H}_{\hat{n}}$ .

**Lemma 19.** An SU(2)-invariant operation on  $\mathcal{H}_{\hat{n}}$  that takes pure states to pure states admits a Kraus decomposition  $\{K_{J,\alpha}\}$ , of the form

$$K_{J,\alpha} = S_{-J} \tilde{K}_{J,\alpha}$$

where  $\tilde{K}_{J,\alpha} = \sum_j c_j^{(J,\alpha)} |j, j\rangle \langle j, j|$  changes the relative amplitudes of the  $|j, j\rangle$  states, possibly eliminating some, and  $S_{-J} = \sum_{j \geq J} |j - J, j - J\rangle \langle j, j|$  shifts the value of  $j$  downward by  $J$ . The coefficients satisfy  $\sum_{J \leq j} \sum_{\alpha} |c_j^{(J,\alpha)}|^2 \leq 1$  for all  $j$ , with equality if the operation is trace-preserving.

Define the weights on  $j$  of the source state  $|\psi\rangle$  and target state  $|\phi\rangle$  by

$$\begin{aligned} p_j &\equiv \langle \psi | \Pi_j | \psi \rangle \\ q_j &\equiv \langle \phi | \Pi_j | \phi \rangle \end{aligned}$$

where  $\Pi_j$  is the projector onto  $|j, j\rangle_{\hat{n}}$ , and define the  $j$ -spectrum by

$$\text{j-Spec}(\psi) \equiv \{j | p_j \neq 0\}.$$

**Theorem 20.** The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by a deterministic SU(2)-invariant operation if and only if

$$p_j = \sum_J w_J q_{j+J}, \quad (6)$$

where the sum is over  $J \in \{0, 1/2, 1, \dots\}$  and where  $0 \leq w_k \leq 1$  and  $\sum_k w_k = 1$ .

**Theorem 21.** The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by stochastic SU(2)-invariant operations if and only if

$$\exists J \in \{0, 1/2, 1, \dots\} : \text{j-Spec}(\phi) \subset \text{j-Spec}(\psi) - J. \quad (7)$$

**Theorem 22.** If there is only a single value of  $J$  such that the condition  $\text{j-Spec}(\phi) \subset \text{j-Spec}(\psi) - J$  holds, then the maximum probability of achieving the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  using SU(2)-invariant operations is

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \min_j \left( \frac{p_j}{q_{j-J}} \right).$$

Finally, defining a gapless  $j$ -spectrum to be one wherein the increment between every successive pair of  $j$  values in the spectrum is 1, we can state our result

**Theorem 24.** In the set of pure resource states for the  $SU(2)$ -SSR that have gapless  $j$ -spectra, the maximum rate at which  $n$  copies of  $|\psi\rangle$  can be converted to  $m$  copies of  $|\phi\rangle$  is determined by a *pair* of measures: the scaled  $j$ -mean

$$\mathcal{M}(|\psi\rangle) \equiv 2\langle\psi|\mathcal{J}|\psi\rangle, \quad (8)$$

and the scaled  $j$ -variance,

$$V(|\psi\rangle) \equiv 4[\langle\psi|\mathcal{J}^2|\psi\rangle - \langle\psi|\mathcal{J}|\psi\rangle^2], \quad (9)$$

where

$$\mathcal{J} \equiv \sum_{j=0, \frac{1}{2}, 1, \dots} j|j, j\rangle\langle j, j|, \quad (10)$$

and the scaling factors were chosen such that  $(|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$  has  $j$ -mean and  $j$ -variance of 1. This rate is given by

$$\lim_{n \rightarrow \infty} \frac{m}{n} = \min \left\{ \frac{\mathcal{M}(|\psi\rangle)}{\mathcal{M}(|\phi\rangle)}, \frac{V(|\psi\rangle)}{V(|\phi\rangle)} \right\}. \quad (11)$$

**Corollary 25.** Asymptotic reversible interconversion is possible within the set of pure resource states for the  $SU(2)$ -SSR that have gapless  $j$ -spectra and that have the same ratio of scaled  $j$ -mean  $\mathcal{M}$  to scaled  $j$ -variance  $V$ . Within each such set, the unique asymptotic measure of  $SU(2)$ -frameness (modulo normalization) is

$$F^\infty(|\psi\rangle) = V(|\psi\rangle). \quad (12)$$

## II. THE RESTRICTION OF LACKING A REFERENCE FRAME

### A. Preliminaries

Reference frames are implicit in the definition of quantum states. For instance, to assert that the quantum state is an eigenstate of angular momentum along the  $\hat{z}$  direction is to describe the state relative to some physical system—a reference frame—that defines the  $\hat{z}$  direction. Different degrees of freedom require different reference frames and are characterized by the group under which they transform. For instance, if the degree of freedom is orientation in space, then the group is  $SO(3)$  and the requisite frame is a triad of orthogonal spatial axes – a Cartesian frame.

If a party lacks a reference frame for some degree of freedom, then they are effectively restricted in the sorts of states they can prepare and the sorts of operations they can implement. Without access to a Cartesian frame, for example, there is nothing with respect to which rotations can be defined, and consequently rotations cannot

be implemented. Similarly, the only states that can be prepared are those that are invariant under rotations. Consequently, any quantum state that is *not* rotationally invariant is a resource. More generally, any system that is known to be aligned with some reference frame is a resource to someone who does not have access to that reference frame. Such a resource, or quantum reference frame, is typically useful as a substitute for a classical reference frame (although perhaps a poor one). For instance, it may allow one to implement, with some probability, the sorts of operations and measurements that one could implement if one had a classical reference frame. Indeed, a non-invariant state stands in for a classical reference frame in a manner similar to the way in which an entangled state can stand in for a quantum channel [6] (although there are differences). Our task here will be to characterize the manner in which such resource states can be transformed under the allowed operations.

We presently sketch the precise nature of this restriction in the general case. Suppose  $G$  denotes the group of transformations associated with the reference frame. The states that can be prepared without access to the frame are those that are invariant under these transformations. Assuming the system is described by a density operator  $\rho \in \mathcal{B}(\mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space and  $\mathcal{B}(\mathcal{H})$  denotes the bounded linear operators on this space, and assuming that  $T : G \rightarrow \mathcal{B}(\mathcal{H})$  denotes the unitary representation of  $G$  that corresponds to the physical transformations in question, the states that can be prepared satisfy

$$T(g)\rho T^\dagger(g) = \rho, \quad \forall g \in G, \quad (13)$$

Equivalently,

$$[\rho, T(g)] = 0, \quad \forall g \in G. \quad (14)$$

Such a state is said to be  $G$ -invariant.

If the system is composed of many subsystems,  $\mathcal{H} = \bigotimes_k \mathcal{H}_k$ , where the  $k$ th subsystem transforms according to the defining representation  $T_k$ , then  $T$  is the tensor product representation of  $G$ , that is,  $T(g) = \bigotimes_k T_k(g)$ .

This restriction on states is sometimes referred to as a *superselection rule* (SSR). Although the latter has often been considered to be an axiomatic restriction rather than arising from the lack of a reference frame, the mathematical characterization is the same. Consequently, we will refer to the restriction of lacking a reference frame for a group  $G$  as a  $G$ -SSR. (Note that the most common conception of a SSR, forbidding coherence between distinguished subspaces, is only appropriate for Abelian groups. For nonAbelian groups, it is more complicated.)

The operations that can be performed under the  $G$ -SSR are those associated with  $G$ -invariant CP maps. Let  $\mathcal{T}$  be the unitary representation of  $G$  on the space of superoperators that corresponds to the physical transformations in question, so that  $\mathcal{T}(g)[X] = T(g)XT^\dagger(g)$ . A CP map  $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is  $G$ -invariant if it satisfies

$$\mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}^\dagger(g) = \mathcal{E}, \quad \forall g \in G, \quad (15)$$

where  $\mathcal{A} \circ \mathcal{B}[\rho] = \mathcal{A}[\mathcal{B}[\rho]]$  denotes a composition of operations, and  $\mathcal{F}^\dagger$ , the Hermitian adjoint for superoperators, is defined by  $\text{Tr}(X\mathcal{F}[Y]) = \text{Tr}(\mathcal{F}^\dagger[X]Y)$  for all  $X, Y \in \mathcal{B}(\mathcal{H})$ . Equivalently,  $\mathcal{E}$  is  $G$ -invariant if it satisfies

$$[\mathcal{E}, \mathcal{T}(g)] = 0, \quad \forall g \in G,$$

where  $[\mathcal{A}, \mathcal{B}] = \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$  is the superoperator commutator.

It is useful to highlight two ways in which the restriction of lacking a reference frame may arise. On the one hand, a party may fail to possess any system that can serve as a reference frame. Such a restriction is difficult to imagine in the case of a Cartesian frame, since all that is required is a system that can define a triad of orthogonal vectors, and these are ubiquitous (although even in this context, achieving high degrees of precision and stability is a challenge). Such a restriction is, however, easy to imagine in the case of more exotic reference frames. For instance, a Bose-Einstein condensate acts as a reference frame for the phase conjugate to atom number, and a superconductor acts as a reference frame for the phase conjugate to number of Cooper pairs [29], and neither is straightforward to prepare.

The other way in which a superselection rule may arise is if a party has a local reference frame, but it is uncorrelated with the reference frame with respect to which the system is ultimately described. An example serves to illustrate the idea. Suppose that two parties, Alice and Charlie, each have a reference frame for the degree of freedom in question, but that these are uncorrelated. If  $g \in G$  is the group element describing the passive transformation from Alice's to Charlie's frame, the absence of correlation amounts to assuming that  $g$  is completely unknown. It follows that if Alice prepares a state  $\rho$  on  $\mathcal{H}$  relative to her frame, the system is represented relative to Charlie's frame by the state <sup>2</sup>

$$\mathcal{G}[\rho] \equiv \int_G dg \mathcal{T}(g) \rho \mathcal{T}^\dagger(g). \quad (16)$$

where  $dg$  is the group-invariant (Haar) measure. We have assumed that  $G$  is a compact Lie group. If  $G$  is instead a finite group, we simply replace  $\int_G dg$  with  $|G|^{-1} \sum_{g \in G}$  where  $|G|$  denotes the order of  $G$ . We call the operation  $\mathcal{G}$  the “ $G$ -twirling” operation. If we are only interested in describing Alice's systems relative to Charlie's reference frame (perhaps because we are only interested in measurements performed relative to the latter), then we can group the states into equivalence classes, where the equivalence relation is equality under  $G$ -twirling. Every equivalence class has a  $G$ -invariant member, satisfying

$\rho = \mathcal{G}[\rho]$ . Indeed, the totality of what Alice can predict about the outcomes of Bob's measurements is always characterized by some  $G$ -invariant density operator, so, relative to Charlie's frame, the density operators satisfying Eq. (13) are the only ones that Alice can prepare.

Similarly, if Alice implements an operation  $\mathcal{E}$  relative to her frame, then relative to Charlie's frame she has implemented the operation

$$\mathfrak{G}(\mathcal{E}) \equiv \int_G dg \mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g^{-1}). \quad (17)$$

Equation (17) has the form of Eq. (16) except with operators replaced by superoperators. It is therefore appropriate to refer to the map  $\mathfrak{G}$  as “super- $G$ -twirling”. If we again choose to always represent operations by Alice relative to the reference frame of Charlie, then all operations are of the form  $\mathcal{E} = \mathfrak{G}(\mathcal{E})$ , and any such operation satisfies Eq. (15).

## B. Kraus representation for $G$ -invariant operations

We now proceed to derive an important result concerning the Kraus representation of  $G$ -invariant operations. Suppose the operation  $\mathcal{E}$  has an  $N$ -term Kraus decomposition  $\{K_\mu\}$ . It is then clear that the operation  $\mathcal{T}(g) \circ \mathcal{E} \circ \mathcal{T}(g^{-1})$  has an  $N$ -term Kraus decomposition  $\{K'_\mu\}$  where  $K'_\mu = \mathcal{T}(g) K_\mu \mathcal{T}^\dagger(g) = \mathcal{T}(g)[K_\mu]$ . But now recall that if two CP maps are equivalent, then the Kraus operators of one are a unitary remixing of those of the other [30]. Equation (15) then implies that there exists an  $N \times N$  unitary matrix  $u(g)$  such that

$$\mathcal{T}(g)[K_\mu] = \sum_{\mu'} u_{\mu\mu'}(g) K_{\mu'}, \quad \forall g \in G. \quad (18)$$

If the Kraus operators are linearly independent (so that the Kraus decomposition has the minimal number of elements), then  $u$  is a unitary representation of  $G$ . The reason is as follows. Suppose that  $\{W_\mu\}$  constitutes a dual basis to  $\{K_\mu\}$  on the operator space  $\mathcal{B}(\mathcal{H})$ , so that the elements of one are orthonormal to those of the other relative to the Hilbert-Schmidt inner product, that is,  $\text{Tr}(W_\mu^\dagger K_{\mu'}) = \delta_{\mu\mu'}$ . It is always possible to find such a dual basis if the  $\{K_\mu\}$  are linearly independent. It follows that

$$\text{Tr}(W_{\mu''}^\dagger \mathcal{T}(g)[K_\mu]) = u_{\mu\mu''}(g), \quad \forall g \in G,$$

and consequently that  $u(g)$  is simply a matrix representation of the superoperator  $\mathcal{T}(g)$ . Because  $\mathcal{T}$  is a representation of  $G$ , so is  $u$ .

By virtue of the unitary freedom in the Kraus decomposition, it is always possible to choose the Kraus operators such that  $u(g)$  is in block-diagonal form, with the blocks labeled by the irreducible representations (irreps) of  $G$  and possibly a multiplicity index, and with

<sup>2</sup> The invariant measure is chosen using the maximum entropy principle: because Charlie has no prior knowledge about Alice's reference frame, he should assume a uniform measure over all possibilities.



the dimensionality of each block corresponding to the dimensionality of the associated irrep. We summarize this result in the following lemma.

**Lemma 1.** *A  $G$ -invariant operation admits a Kraus decomposition with Kraus operators  $K_{jm\alpha}$ , where  $j$  denotes an irrep,  $m$  a basis for the irrep, and  $\alpha$  a multiplicity index, satisfying*

$$\mathcal{T}(g)[K_{jm\alpha}] = \sum_{m'} u_{mm'}^{(j)}(g) K_{jm'\alpha}, \quad \forall g \in G, \quad (19)$$

where  $u^{(j)}$  is an irreducible unitary representation of  $G$ .

Notice that the action of the group only mixes Kraus operators associated with the same  $j$  and  $\alpha$ . The set of operators  $\{K_{jm\alpha}|m\}$  for fixed  $j$  and  $\alpha$  is called an *irreducible tensor operator* [17] of rank  $j$  in nuclear and atomic physics.<sup>3</sup>

From Eq. (19), it is clear that the  $K_{j,m,\alpha}$  play the same role in the Hilbert-Schmidt operator space  $\mathcal{B}(\mathcal{H})$  as the joint eigenstates  $|j, m, \beta\rangle$  of  $J^2$  and  $J_z$  play in the Hilbert space  $\mathcal{H}$ .

A  $G$ -invariant operation which is of the form  $\mathcal{E}_{j,\alpha}(\cdot) = \sum_m K_{jm\alpha}(\cdot) K_{jm\alpha}^\dagger$  where the  $K_{jm\alpha}$  satisfy Eq. (19) for some irrep  $j$  and multiplicity index  $\alpha$  will be called an *irreducible  $G$ -invariant operation*. Every  $G$ -invariant operation is clearly a sum of irreducible  $G$ -invariant operations.

An obvious question to ask at this stage is whether it is always possible to physically implement any given  $G$ -invariant operation. That it *is* possible is guaranteed by an application of the Stinespring dilation theorem to  $G$ -invariant operations [31]. The theorem ensures that it suffices to prepare a  $G$ -invariant state of an ancilla, couple this to the system via a  $G$ -invariant unitary, and then implement a  $G$ -invariant measurement upon the ancilla.

### C. Frameness monotones

There are three sorts of frameness monotones that we will consider in this work. We term these deterministic, ensemble and stochastic frameness monotones. We consider each in turn.

We define a *deterministic  $G$ -frameness monotone* as a function  $F : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$  that does not increase under deterministic  $G$ -invariant operations. Specifically,  $F$  is a  $G$ -frameness monotone if for all  $\rho \in \mathcal{B}(\mathcal{H})$  and for all trace-preserving CP maps  $\mathcal{E}$  satisfying Eq. (15),

$$F(\mathcal{E}(\rho)) \leq F(\rho). \quad (20)$$

This definition is in analogy with that of a type 2 entanglement monotone, introduced in Ref. [7], which is a

function  $E : \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \dots) \rightarrow \mathbb{R}^+$  that is non-increasing under deterministic LOCC operations. The notion of a deterministic frameness monotone was first made explicit in Appendix A of Ref. [11].

We define an *ensemble  $G$ -frameness monotone* as a function  $F : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$  that does not increase *on average* under  $G$ -invariant operations. This definition is in analogy with that of standard entanglement monotones, functions  $E : \mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \dots) \rightarrow \mathbb{R}^+$  that are non-increasing on average under LOCC operations [32, 33]. To make the definition explicit, we note that the most general sorts of  $G$ -invariant operations include: (1)  $G$ -invariant measurements, and (2) forgetting information.  $G$ -invariant measurements generate a transformation from a state  $\rho$  to an ensemble  $\{(w_\mu, \sigma_\mu)\}$  (i.e.  $\rho$  collapses to  $\sigma_\mu$  with probability  $w_\mu$ ) where  $w_\mu \sigma_\mu = \mathcal{E}_\mu(\rho)$  for some trace-nonincreasing  $G$ -invariant operation  $\mathcal{E}_\mu$ . For a frameness monotone  $F$  to be nonincreasing on average, we require

$$\sum_\mu w_\mu F(\sigma_\mu) \leq F(\rho). \quad (21)$$

The second requirement is that if one knows the state to be  $\sigma_\mu$  with probability  $w_\mu$  and then discards the information about  $\mu$ , resulting in the state  $\sigma = \sum_\mu w_\mu \sigma_\mu$ , then  $F$  is nonincreasing,

$$F(\sigma) \leq \sum_\mu w_\mu F(\sigma_\mu). \quad (22)$$

Note that any non-decreasing concave function of an ensemble frameness monotone is also an ensemble frameness monotone. To see that this is the case, consider the  $G$ -invariant transformation  $\rho \rightarrow \{(w_\mu, \sigma_\mu)\}$  and let  $f : [0, 1] \rightarrow \mathbb{R}$  be a non-decreasing concave function (i.e.  $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$  for all  $t, x, y \in [0, 1]$ ). It is then straightforward to see that if  $F$  is an ensemble frameness monotone, so that  $F(\rho) \geq \sum_\mu w_\mu F(\sigma_\mu)$ , then  $f(F(\rho)) \geq \sum_\mu w_\mu f(F(\sigma_\mu))$ .

The idea of requiring that a measure of frameness be nonincreasing on average under invariant operations is present in Vaccaro *et al.* [8] and Schuch *et al.* [10].

Finally, we define a *stochastic  $G$ -frameness monotone* as a  $G$ -frameness monotone that is nonincreasing even under stochastic (that is, nondeterministic)  $G$ -invariant operations. Specifically,  $F$  is a stochastic  $G$ -frameness monotone if for all  $\rho \in \mathcal{B}(\mathcal{H})$  and for all  $\sigma \in \mathcal{B}(\mathcal{H})$  such that the transformation  $\rho \rightarrow \sigma$  can be achieved either deterministically or indeterministically by a  $G$ -invariant operation,  $F(\sigma) \leq F(\rho)$ . Equivalently, for all  $\rho \in \mathcal{B}(\mathcal{H})$  and for all trace-nonincreasing completely positive maps  $\mathcal{S}$  that are  $G$ -invariant, we require that

$$F(\mathcal{S}(\rho)/\text{Tr}(\mathcal{S}(\rho))) \leq F(\rho). \quad (23)$$

An example from entanglement theory of a stochastic monotone is the Schmidt number, which can not be increased even with probability less than 1 using LOCC.

<sup>3</sup> We thank Matthias Christandl for bringing this to our attention.

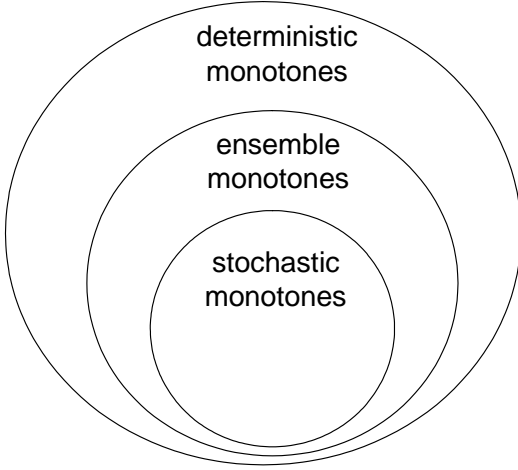


FIG. 1: A Venn diagram of frameness monotones for pure states.

It is sometimes useful to consider a measure of frameness that is only defined on a subset of all states. For instance, it may be defined only on pure states (or only on a subset of pure states). Indeed, this situation will be the norm in the present work. In this case, the condition of Eq. (20) (respectively Eq. (23)) is only required to hold if the map  $\mathcal{E}$  (respectively  $\mathcal{S}$ ) takes states in the subset of interest to others in that subset – otherwise the left-hand side of the condition is not well-defined. Similarly, the condition of Eq. (21) is only required to hold when every outcome of the measurement yields a state in the subset of interest, and that of (22) is only required to hold if the subset is closed under convex combination. These weaker conditions are all that are required to hold for a measure with a restricted domain of definition to be deemed a monotone of each type.

For measures that are only defined on pure states, ensemble monotones are only required to satisfy Eq. (21) (because the pure states are not closed under convex combination). In this case, if a measure is a stochastic frameness monotone then it is an ensemble frameness monotone because Eq. (23) implies  $F(\sigma_\mu) \leq F(\rho)$ , which implies Eq. (21). Furthermore, if a measure is an ensemble frameness monotone then it is a deterministic frameness monotone because deterministic  $G$ -invariant transformations are a special case of  $G$ -invariant measurements wherein there is only a single outcome. These inclusions are denoted schematically in Fig. 1.

There are a couple of other features that are nice for a measure of frameness to have although these are, strictly speaking, only a choice of convention: (1) Positivity,  $F(\rho) \geq 0$  for all  $\rho$  in the domain of definition, and (2) Zero on  $G$ -invariant states  $F(\rho) = 0$  if  $[\rho, T(g)] = 0$  for all  $g \in G$ . Where there is a freedom in the definition of a frameness measure, we will choose conventions ensuring that these features hold.

#### D. The motivation for requiring monotonicity

The motivation for demanding that a measure of the resource be monotonically nonincreasing under the allowed operations is that it is a necessary condition if the measure is to have operational significance. This is an important point that is worth making precise.

We shall say that a measure of a resource is operational if and only if it quantifies the *optimal* figure of merit for some task that requires the resource for its implementation. Specifically, we imagine a task that is described entirely operationally (that is, in terms of empirically observable consequences) and a figure of merit that quantifies the degree of success achieved by every possible protocol for implementing the task (under the restriction that defines the resource theory). Success might be measured in terms of the probability of achieving some outcome, or the yield of some other resource, etcetera. The key point is that *any processing of the resource* (consistent with the restriction that defines the resource theory) cannot increase an operational measure of that resource because the definition of an operational measure *already incorporates* an optimization over protocols and thus an optimization over all such processings.

Because the sorts of operations that can appear in a protocol for the task may be restricted, an operational measure might only be monotonically nonincreasing for a restricted set of operations. The various monotones described above – deterministic, ensemble and stochastic – are appropriate for different sorts of tasks.

Some tasks may be achieved by protocols that at their end yield an ensemble of states  $\{(w_\mu, \sigma_\mu)\}$ . If the figure of merit for the task is an average  $\sum_\mu w_\mu f(\sigma_\mu)$  of some figure of merit  $f$  for the final state, then the *optimal* figure of merit for the task (optimized over all protocols achieving the task) is an ensemble monotone by definition.

If the figure of merit  $f$  is a linear function of the density operator,  $f(\sum_\mu w_\mu \sigma_\mu) = \sum_\mu w_\mu f(\sigma_\mu)$ , then  $f$  is unchanged by forgetting information. Furthermore, the condition of being nonincreasing on average under measurements becomes the condition of being nonincreasing under deterministic operations. Consequently, the notion of a deterministic frameness monotone is only distinct from that of an ensemble frameness monotone for non-linear figures of merit. As an example, if one has a figure of merit over  $\rho$  that quantifies what can be achieved with  $N > 1$  copies of  $\rho$ , then even if the achievement is itself some linear function of  $\rho^{\otimes N}$ , the figure of merit need not be a linear function of  $\rho$ . As noted above, the resource theory for the  $Z_2$ -SSR provides an example of an operationally well-motivated measure of frameness, the asymptotic rate of reversible interconversion of resources, which is a deterministic monotone but not an ensemble monotone.

Other tasks might incorporate post-selection in their definition. Consequently, if the protocol yields an ensemble of outcomes, the figure of merit for the protocol may be the maximum of some figure of merit for each

possible outcome rather than the average. This is the case, for instance, when one is interested in the best-case or worst-case scenarios. The measures of the resource for such tasks satisfy the strongest possible constraint of monotonicity: they must be stochastic monotones.

### E. Single-copy frame manipulations

For each sort of SSR considered in the paper, we seek to find necessary and sufficient conditions for the existence of a deterministic  $G$ -invariant operation that converts a pure state  $|\psi\rangle$  into another  $|\phi\rangle$ . In the context of entanglement theory, these are provided by Nielsen's theorems [15]. If a particular conversion cannot be achieved deterministically, then we wish to know the maximum probability with which it can be achieved. This is the analogue of Vidal's formula in the theory of entanglement [16].

### F. Asymptotic frame manipulations and the unique asymptotic measure of framedness

Even though a single copy of  $\rho$  may not be converted to a single copy of  $\sigma$  deterministically under the  $G$ -SSR, a transformation of  $N$  copies of  $\rho$  to  $M$  copies of  $\sigma$  might still be achievable. Of particular interest is the question of whether the transformation

$$\rho^{\otimes N} \rightarrow \sigma^{\otimes M}$$

can be achieved in the limit  $N \rightarrow \infty$ , in the sense that there exists a deterministic  $G$ -invariant operation  $\mathcal{E}$  such that

$$\text{Fid}(\mathcal{E}(\rho^{\otimes N}), \sigma^{\otimes M}) \simeq 1$$

where  $\text{Fid}(\rho, \sigma) \equiv \text{Tr}|\sqrt{\rho}\sqrt{\sigma}|$  is the fidelity. The maximum ratio of  $M$  to  $N$  in the asymptotic limit,  $R_\sigma(\rho) \equiv \lim_{N \rightarrow \infty} M/N$ , is called the asymptotic rate of conversion of  $\rho$  to  $\sigma$ .

Clearly, the asymptotic rate of conversion to  $\sigma$ ,  $R_\sigma$ , is a deterministic framedness monotone. The proof is by contradiction. If it were not, then there would exist a deterministic  $G$ -invariant operation  $\mathcal{E}$  that could be performed on each of the  $N$  copies of  $\rho$  such that one could then generate copies of  $\sigma$  at an asymptotic rate of  $R_\sigma(\mathcal{E}(\rho)) \geq R_\sigma(\rho)$ , contradicting the assumption that  $R_\sigma$  quantified the optimal rate. An analogue of this result holds in any resource theory.

Note also that  $R_\sigma$  is weakly additive,

$$R_\sigma(\rho^{\otimes 2}) = 2R_\sigma(\rho).$$

The proof is simply that  $N$  copies of  $\rho^{\otimes 2}$  are equivalent to  $2N$  copies of  $\rho$  and consequently can yield twice as many copies of  $\sigma$ .

The resource theory that arises from a restriction on operations is particularly simple if any form of the resource can be *reversibly* transformed (in an asymptotic sense) to any other form under the restricted operations. In this case, for any pair of states,  $\rho$  and  $\sigma$ , one can reversibly transform  $N$  copies of  $\rho$  into  $M$  copies of  $\sigma$  (or a good approximation thereof) in the limit of large  $N$ . That is,

$$\rho^{\otimes N} \Leftrightarrow \sigma^{\otimes M}, \quad (24)$$

in the sense that there exist  $G$ -invariant operations  $\mathcal{E}$  and  $\mathcal{E}'$  such that

$$\begin{aligned} \text{Fid}(\mathcal{E}(\rho^{\otimes N}), \sigma^{\otimes M}) &\simeq 1 \\ \text{Fid}(\mathcal{E}'(\sigma^{\otimes M}), \rho^{\otimes N}) &\simeq 1 \end{aligned}$$

in the limit  $N \rightarrow \infty$ .

If there exist such asymptotic reversible transformations between any two states then a single measure of  $G$ -framedness over the states is sufficient to characterize the rate of interconversion between any two. Specifically, if  $\rho^{\otimes N} \Leftrightarrow \sigma^{\otimes M}$ , then we can define a measure of  $G$ -framedness over all states,  $F^\infty$ , by

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{F^\infty(\rho)}{F^\infty(\sigma)}. \quad (25)$$

This clearly does not fix the normalization of  $F^\infty$ , however, a useful convention for doing so is to choose a particular state  $\sigma$  to be the “standard” against which all others are compared and to set  $F^\infty(\sigma) = 1$  for this state.

One of the most celebrated results in the theory of entanglement is that there is a unique measure of entanglement for bipartite pure states, the entropy of entanglement, which quantifies the number of e-bits (i.e. maximally entangled states of two qubits) that can be distilled from a given pure state  $|\psi\rangle$  in the asymptotic limit of many copies [32, 33]. We show that whether one can obtain a unique measure of framedness for pure states depends on the nature of the group associated with the frame. In particular, a unique measure arises for the pure states under the  $Z_2$ -SSR, but asymptotically reversible transformations exist only for certain subsets of pure states for the  $U(1)$ -SSR and the  $SU(2)$ -SSR.

If there is a unique measure of framedness  $F^\infty$ , then it is a deterministic monotone and is weakly additive. This follows from the fact that such a measure is an instance of an asymptotic conversion rate and the fact that such rates are deterministic monotones and are weakly additive (as shown above). As it turns out, however,  $F^\infty$  need not be an ensemble framedness monotone. A counterexample is provided by the resource theory for the  $Z_2$ -SSR. This result is particularly interesting because it has no analogue in pure state bipartite entanglement theory: the entropy of entanglement, which quantifies the asymptotic rate of reversible interconversion between entangled states, is an ensemble monotone. To the authors' knowledge, it is an open question whether there exist subsets

of the mixed bipartite entangled states or multipartite entangled states that exhibit similar behaviour, namely, that any two states in the subset can be reversibly interconverted asymptotically but the rate of interconversion is not an ensemble monotone.

### III. RESOURCE THEORY OF THE U(1)-SSR

#### A. Phase references

The first example we consider is that of a *phase reference*, for which the relevant group of transformations is U(1), the group of real numbers modulo  $2\pi$  under addition. One requires a phase reference, for instance, to prepare a coherent state of the electromagnetic field. The phase reference typically takes the form of a strong classical field (a local oscillator) with respect to which the phase of the coherent state is defined. For two parties to share a phase reference, their local oscillators must have a well-known relative phase, which is to say that they must be phase-locked.

A phase reference is also required to prepare coherent superpositions of eigenstates of any additively conserved charge. A charge operator differs from a number operator because there is no lower bound on its spectrum. In what follows, we shall presume a phase conjugate to number rather than charge, although the results could easily be adapted to the case of charge.

Finally, note that if one possesses a reference frame consisting of a single direction in space – the frame relative to which a system can be described as pointing up or down – then what one lacks to achieve a full Cartesian frame (a triad of orthogonal directions) is a phase reference. In this sense, the lack of a phase reference is a milder restriction than the lack of a full Cartesian frame.

A phase shift of  $\phi \in (0, 2\pi)$  is represented by the unitary

$$T(\phi) = e^{i\phi\hat{N}},$$

where  $\hat{N}$  is the number operator.  $T$  is a unitary representation of U(1). The states that are U(1)-invariant (or “phase-shift-invariant”) are those satisfying

$$T(\phi)\rho T^\dagger(\phi) = \rho \quad \forall \phi \in \text{U}(1).$$

This is equivalent to the condition

$$[\rho, \hat{N}] = 0,$$

so that the invariant  $\rho$  are block-diagonal relative to the eigenspaces of  $\hat{N}$ .

It will be useful for us to decompose the Hilbert space  $\mathcal{H}$  into a direct sum of the carrier spaces  $\mathcal{H}_n$  for the irreducible representations of U(1), that is, the eigenspaces of the total number operator  $\hat{N}$ ,

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \quad (26)$$

The dimensionality of each  $\mathcal{H}_n$  is simply the multiplicity of the  $n$ th irreducible representation of U(1) on the system. As an example, if our system is  $K$  optical modes, then  $\mathcal{H}_n$  is the eigenspace of states containing  $n$  photons and has dimension equal to the number of ways of distributing these  $n$  photons among  $K$  modes.

Let  $\beta$  be a multiplicity index, so that we may denote a basis for  $\mathcal{H}_n$  by  $|n, \beta\rangle$ . An arbitrary state can then be written as

$$|\psi\rangle = \sum_{n,\beta} c_{n,\beta} |n, \beta\rangle,$$

and transforms under phase shifts as

$$T(\phi)|\psi\rangle = \sum_n e^{in\phi} \sum_\beta c_{n,\beta} |n, \beta\rangle. \quad (27)$$

In this article, we will be considering the resource theory for pure states only. From Eq. (27), it is clear that all and only those pure states that are confined to a single  $\mathcal{H}_n$  ( $c_{n,\beta} \neq 0$  for only a single value of  $n$ ) are U(1)-invariant and thus preparable under the U(1)-SSR. In the theory of entanglement any state that cannot be prepared by LOCC can be considered a resource. In our case, any state that cannot be prepared by U(1)-invariant operations is considered a resource. The one-mode state  $|0\rangle$  or  $|1\rangle$  or the two-mode state  $(a|01\rangle + b|10\rangle)/\sqrt{2}$  are not resources because they can be prepared under the U(1)-SSR (i.e. they are considered cheap). On the other hand, the one-mode state  $a|0\rangle + b|1\rangle$  or the two mode state  $a|01\rangle + b|12\rangle$  *cannot* be prepared under the U(1)-SSR and therefore *do* constitute resources.

Because the multiplicity space carries a trivial representation of U(1) (phase shifts act as identity upon it), it is clear that any change to the multiplicity index does not require a phase reference. In other words, any operation *within* one of the  $\mathcal{H}_n$  is possible under the U(1)-SSR. Consequently, any pure state  $|\psi\rangle = \sum_{n,\beta} a_{n,\beta} |n, \beta\rangle$  can be taken, by a U(1)-invariant unitary operation, to the form

$$|\psi\rangle = \sum_n a_n |n\rangle, \quad (28)$$

where  $|n\rangle$  is some particular element of  $\mathcal{H}_n$ . We will presume this form for states in what follows. We are thereby restricting ourselves to the subspace  $\mathcal{H}' = \text{span}\{|n\rangle\} \subseteq \mathcal{H}$ . In the optical context, for example, this corresponds to transforming all multi-mode states into single-mode states. The analogue of this convention in the context of entanglement theory for pure bipartite states would be to work with a particular choice of Schmidt basis. Because the transformation from one Schmidt basis to another can always be achieved by local unitaries, the Schmidt basis is irrelevant to the question of entanglement manipulation and it is therefore convenient to factor it out of the problem.

Relative to this standard form, the resource states are simply those for which  $a_n \neq 0$  for more than one value of

$n$ . To see how such resource states can be manipulated, we must determine what can be achieved using  $U(1)$ -invariant operations.

### B. $U(1)$ -invariant operations

We now apply Lemma 1 to the characterization of  $U(1)$ -invariant operations. Note first that the irreducible representations of  $U(1)$  are labeled by an integer  $k$ , and are all 1-dimensional. The  $k$ th irreducible representation  $u_k : U(1) \rightarrow \mathbb{C}$  has the form

$$u_k(\phi) = e^{-ik\phi}.$$

It follows that the Kraus operators  $K_{k,\alpha}$  of a  $U(1)$ -invariant operation are labeled by an irrep  $k$  and a multiplicity index  $\alpha$  and satisfy

$$e^{i\phi\hat{N}} K_{k,\alpha} e^{-i\phi\hat{N}} = e^{ik\phi} K_{k,\alpha}, \quad \forall \phi \in U(1). \quad (29)$$

Note that by virtue of the fact that the irreps are 1d, the Kraus operators do not get mixed with one another under the action of  $U(1)$ . This provides a significant simplification relative to the non-Abelian case.

As we are confining ourselves to the subspace  $\mathcal{H}' = \text{span}\{|n\rangle\}$ , the most general expression for  $K_{k,\alpha}$  is

$$K_{k,\alpha} = \sum_{n,n'} c_{nn'}^{(k,\alpha)} |n\rangle\langle n'|, \quad (30)$$

where the  $c_{nn'}^{(k,\alpha)}$  are complex coefficients. Plugging this into Eq. (29) yields the constraint

$$\sum_{n,n'} c_{nn'}^{(k,\alpha)} e^{i(n-n')\phi} |n\rangle\langle n'| = \sum_{n,n'} c_{nn'}^{(k,\alpha)} e^{ik\phi} |n\rangle\langle n'|,$$

from which it follows that  $n' = n - k$  and consequently

$$K_{k,\alpha} = \sum_{n=\max\{0,k\}}^{\infty} c_n^{(k,\alpha)} |n\rangle\langle n-k|, \quad (31)$$

for some amplitudes  $c_n^{(k,\alpha)}$ . (Note that if we were considering a phase degree of freedom conjugate to charge rather than number, the sum would have no lower bound.)

In order for the operation to be trace-nonincreasing, we require  $\sum_{k,\alpha} K_{k,\alpha}^\dagger K_{k,\alpha} \leq I$ , which implies that  $\sum_{k,\alpha} |c_n^{(k,\alpha)}|^2 \leq 1$  for all  $n$ , where the inequalities are saturated if the operation is trace-preserving.

We summarize this result in the following lemma, where we also introduce a useful factorization for the Kraus operators.

**Lemma 2.** *An arbitrary  $U(1)$ -invariant operation admits a Kraus decomposition  $\{K_{k,\alpha}\}$ , where  $k$  and  $\alpha$  are integers, such that*

$$K_{k,\alpha} = S_k \tilde{K}_{k,\alpha} \quad (32)$$

where  $\tilde{K}_{k,\alpha} = \sum_n c_n^{(k,\alpha)} |n\rangle\langle n|$  changes the relative amplitudes of the different number states, possibly eliminating some, and  $S_k = \sum_{n=\max\{0,-k\}} |n+k\rangle\langle n|$  shifts the number of excitations upward by  $k$ , that is, upward by  $|k|$  if  $k > 0$ , and downward by  $|k|$  if  $k < 0$ . The coefficients satisfy  $\sum_{k,\alpha} |c_n^{(k,\alpha)}|^2 \leq 1$  for all  $n$ , with equality if the operation is trace-preserving.

As was mentioned in Sec. II B, the Stinespring dilation theorem implies that there is always a way of physically implementing any  $U(1)$ -invariant operation. Nonetheless, it is worth saying a few words about how this is achieved. Just as the restriction of LOCC still permits one to add and discard local ancillae for free, in the resource theory for a  $U(1)$ -SSR, one can add and discard ancillae prepared in  $U(1)$ -invariant states for free. In order to shift the number of the system up by  $k$  (i.e. to implement the operation  $S_k(\cdot)S_k^\dagger$ ), one simply adds an ancilla in an eigenstate  $|k\rangle$  of the number operator and implements the  $U(1)$ -invariant unitary operation that transforms the two-mode state  $|n\rangle|k\rangle$  into the one mode state  $|n+k\rangle$ . To shift the number down by  $k$ , one simply implements the  $U(1)$ -invariant unitary operation that takes the one-mode state  $|n+k\rangle$  to the two-mode state  $|n\rangle|k\rangle$ , and then discards the second mode. This sort of argument was used in Schuch *et al.* [10] to justify Eq. (31).

#### 1. $U(1)$ -invariant unitaries

As discussed above, all unitary operations within a given subspace  $\mathcal{H}_n$  are  $U(1)$ -invariant. However, there are more  $U(1)$ -invariant unitaries besides these, specifically, nontrivial unitaries on the subspace  $\mathcal{H}' = \text{span}\{|n\rangle\}$ . Because unitary operations have a single Kraus operator, they are irreducible  $U(1)$ -invariant operations. However, the only way in which a single Kraus operator  $K$  can be unitary is if  $k = 0$  in Eq. (32), i.e. the operation does not allow shifts in the number, and  $|c_n| = 1$ , so that  $K$  must have the form  $\sum_n e^{i\chi_n} |n\rangle\langle n|$ . All told, the unitary operations that are  $U(1)$ -invariant have the effect of merely changing the relative phases of the  $|n\rangle$ .

At first glance, it might seem surprising that the phase of a state can be changed without requiring a phase reference. Perhaps the easiest way to develop an intuition for why this is true is to consider two parties who don't share a phase reference. If Alice and Bob share only a notion of what is up, that is, the  $\hat{z}$  axis of a Cartesian frame, then what they are lacking, relative to a full Cartesian frame, is the angle between their local  $\hat{x}$  axes. This scenario is an example of lacking a phase reference. Alice certainly cannot prepare a state of definite phase relative to Bob's frame, nor gain any information about this phase, because this requires sharing a common  $\hat{x}$  axis with Bob. However, she *can* change the phase of a state relative to Bob's frame by a fixed amount because this only requires performing a rotation about the common  $\hat{z}$  axis.

By a  $U(1)$ -invariant unitary, any state  $|\psi\rangle = \sum_n a_n |n\rangle$  can be taken to the form

$$|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle, \quad (33)$$

where  $\sum_n p_n = 1$ , that is, a form with real-amplitude coefficients. Consequently, to understand the possible resource manipulations, it suffices to consider states of this standard form – the real-amplitude states on  $\mathcal{H}' = \text{span}\{|n\rangle\} \subseteq \mathcal{H}$ . This convention is analogous, in the entanglement theory of pure bipartite states, to restricting attention not just to states with a fixed Schmidt basis but with real-amplitude Schmidt coefficients (because the phases of the Schmidt coefficients can be changed by local unitaries).

### C. Deterministic single-copy transformations

Consider first the question of which transformations  $|\psi\rangle \rightarrow |\phi\rangle$  can be achieved deterministically using only  $U(1)$ -invariant operations. We assume the states to be in the standard form,  $|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle$  and  $|\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$ , and we denote the vector with components  $q_n$  by  $\vec{q}$ . We also define a shift operator  $\Upsilon_k$  on this vector space by  $\Upsilon_k \vec{q} = \vec{q}'$  where  $q'_{n+k} = q_n$ .

**Theorem 3.** *The necessary and sufficient conditions for the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  to be possible by a deterministic  $U(1)$ -invariant operation is if  $\vec{p}$  can be obtained from  $\vec{q}$  by a convex sum of shift operations, that is,*

$$\vec{p} = \sum_{k=-\infty}^{\infty} w_k \Upsilon_k \vec{q}, \quad (34)$$

where  $0 \leq w_k \leq 1$  and  $\sum_k w_k = 1$ .

**Proof.**<sup>4</sup> To be  $U(1)$ -invariant, the operation must have Kraus operators  $\{K_{k,\alpha}\}$  of the form specified in lemma 2. Given that the operation implements a pure-to-pure transformation, each Kraus operator must take  $|\psi\rangle$  to the same state, that is, for all  $k, \alpha$ ,

$$K_{k,\alpha} |\psi\rangle = \sqrt{w_{k,\alpha}} |\phi\rangle. \quad (35)$$

where  $0 \leq w_{k,\alpha} \leq 1$ . However,

$$K_{k,\alpha} |\psi\rangle = \sum_n c_n^{(k,\alpha)} \sqrt{p_n} |n+k\rangle \quad (36)$$

$$= \sqrt{w_{k,\alpha}} \sum_{n'} \sqrt{q_{n'}} |n'\rangle, \quad (37)$$

and therefore

$$(c_n^{(k,\alpha)})^2 p_n = w_{k,\alpha} q_{n+k}. \quad (38)$$

<sup>4</sup> The proofs are not required for the intelligibility of the text and we recommend that they be ignored on a first reading.

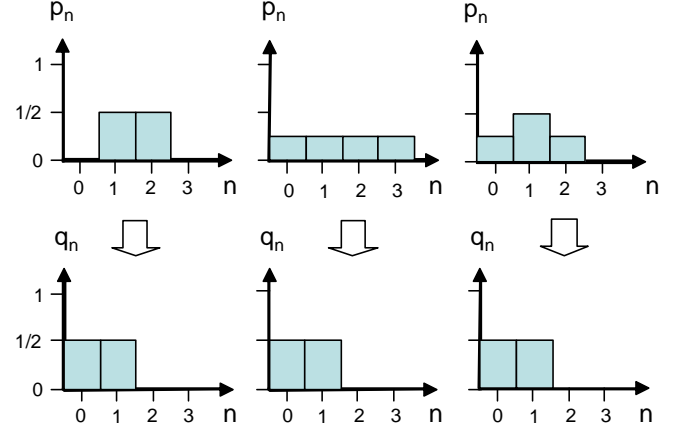


FIG. 2: Three examples of transformations that can be achieved by deterministic  $U(1)$ -invariant operations.

For the transformation to be deterministic, we require that  $\sum_{k,\alpha} \langle \psi | K_{k,\alpha}^\dagger K_{k,\alpha} | \psi \rangle = 1$ , which implies that  $\sum_n \left[ \sum_{k,\alpha} (c_n^{(k,\alpha)})^2 \right] p_n = 1$ , and consequently that  $\sum_{k,\alpha} (c_n^{(k,\alpha)})^2 = 1$  for all  $n$  such that  $p_n \neq 0$ .

Summing Eq. (38) over  $k$  and  $\alpha$  and defining  $w_k \equiv \sum_\alpha w_{k,\alpha}$  yields  $p_n = \sum_k w_k q_{n+k}$ , which, modulo a change in the sign of the dummy variable, is equivalent to Eq. (34).

Conversely, if Eq. (34) holds, then we have  $p_n = \sum_k w_k q_{n+k}$  and we can define a set of amplitudes  $c_n^{(k)} \equiv \sqrt{w_k q_{n+k} / p_n}$  (with  $c_n^{(k)} \equiv 0$  for  $n$  such that  $p_n = 0$ ). It follows that we can define operators  $K_k = S_k \tilde{K}_k$  where  $\tilde{K}_k = \sum_n c_n^{(k)} |n\rangle \langle n|$  are positive operators and where  $\sum_{k,\alpha} \langle \psi | K_{k,\alpha}^\dagger K_{k,\alpha} | \psi \rangle = \sum_n \sum_k (c_n^{(k)})^2 p_n = 1$ . Consequently, the operators  $K_k = S_k \tilde{K}_k$  can constitute the Kraus operators for a  $U(1)$ -invariant operation that is deterministic in its action on  $|\psi\rangle$ . Finally, it is straightforward to verify that this operation achieves the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  QED.

Some examples of transformations that can be achieved deterministically are illustrated in Fig. 2. The first example,  $(|1\rangle + |2\rangle) / \sqrt{2} \rightarrow (|0\rangle + |1\rangle) / \sqrt{2}$ , satisfies the condition because  $\vec{p} = \Upsilon_1 \vec{q}$ . The  $U(1)$ -invariant operation that achieves the transformation has a single Kraus operators  $S_{-1}$  corresponding to a shift of the number downward by 1. (The operation is deterministic because  $S_{-1}^\dagger S_{-1}$  acts as identity on  $(|1\rangle + |2\rangle) / \sqrt{2}$ .) The second example,  $(|0\rangle + |1\rangle + |2\rangle + |3\rangle) / 2 \rightarrow (|0\rangle + |1\rangle) / \sqrt{2}$ , satisfies the condition because  $\vec{p} = \frac{1}{2} \vec{q} + \frac{1}{2} \Upsilon_2 \vec{q}$ , and the  $U(1)$ -invariant operation that achieves the transformation has Kraus operators  $K_0 = |0\rangle \langle 0| + |1\rangle \langle 1|$  and  $K_{-2} = S_{-2} (|2\rangle \langle 2| + |3\rangle \langle 3|)$ , corresponding to implementing a projective-valued measure  $\{|0\rangle \langle 0| + |1\rangle \langle 1|, |2\rangle \langle 2| + |3\rangle \langle 3|\}$  and shifting the number downward by 2 upon obtaining the second outcome. Fi-

nally, the third transformation,  $(|0\rangle + \sqrt{2}|1\rangle + |2\rangle)/2 \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$ , satisfies the condition because  $\vec{p} = \frac{1}{2}\vec{q} + \frac{1}{2}\Upsilon_1\vec{q}$  and the operation has Kraus operators  $K_0 = |0\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 1|$  and  $K_{-1} = S_{-1}\left(\frac{1}{\sqrt{2}}|1\rangle\langle 1| + |2\rangle\langle 2|\right)$  corresponding to a measurement of the POVM  $\{|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|\}$  followed by a shift downward by 1 upon obtaining the second outcome.

The problem of determining whether Eq. (34) is satisfied reduces to determining whether  $\vec{p}$  falls in the convex hull of the  $\Upsilon_k\vec{q}$ . If  $\vec{p}$  has a finite number of nonzero elements, the number of  $k$  values over which one must vary is also finite.

It is worth noting that a necessary condition for the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  to be achieved by a  $U(1)$ -invariant operation is that  $\vec{p}$  is majorized by  $\vec{q}$  (an introduction to the notion of majorization can be found in Ref. [34]). The proof is simply that the shift operation  $\Upsilon_k$  is a type of permutation, and consequently,  $\sum_k w_k \Upsilon_k$  is a doubly-stochastic matrix. Thus if Eq. (34) holds, then  $\vec{p}$  can be obtained from  $\vec{q}$  by a doubly-stochastic matrix, and it then follows from the Polya-Littlewood-Richardson theorem [34] that  $\vec{p}$  is majorized by  $\vec{q}$ .

Majorization is well-known in quantum information theory because one entangled state can be transformed deterministically to another by LOCC if and only if the spectrum of the reduced density operator of the one is majorized by that of the other [15]. In the present context, majorization is a necessary but not a sufficient condition, so the conditions that  $\vec{p}$  and  $\vec{q}$  must satisfy are *stronger* than the conditions that the spectra of the entangled states must satisfy. Only if the doubly stochastic matrix connecting  $\vec{q}$  to  $\vec{p}$  is a convex sum of permutations of a particular type, namely permutations that merely shift each nonzero element of  $\vec{q}$  by the same fixed amount, will the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  be possible.

## D. Stochastic single-copy transformations

### 1. Necessary and sufficient conditions

We now consider the problem of achieving the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  with some non-zero probability, i.e. stochastically rather than deterministically, using only  $U(1)$ -invariant operations. In this case we are able not only to shift the distribution over number rigidly, but also to change the relative probabilities assigned to different number eigenstates. Therefore, the only feature of  $\psi$  and  $\phi$  that is relevant to the question of whether  $|\psi\rangle \rightarrow |\phi\rangle$  under stochastic  $G$ -invariant operations is the set of number eigenvalues to which they assign non-zero probability. If  $|\psi\rangle = \sum_n \sqrt{p_n}|n\rangle$ , then this set for  $|\psi\rangle$  can be specified as  $\{n|p_n \neq 0\}$ . The cardinality of this set will be denoted by  $\mathcal{S}(\psi)$ . It is also useful to list the elements of the set in ascending order, and to denote the

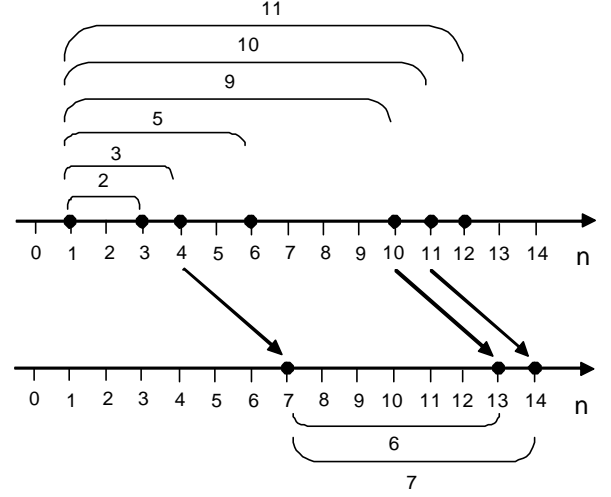


FIG. 3: An example of a transformation that can be achieved by a stochastic  $U(1)$ -invariant operation.

ordered set and its elements by

$$\text{Spec}(\psi) \equiv \{n_1(\psi), n_2(\psi), \dots, n_{\mathcal{S}(\psi)}(\psi)\}.$$

We refer to this set as the *number spectrum* of  $|\psi\rangle$ . As an example, if  $|\psi\rangle = \sqrt{1/2}|0\rangle + \sqrt{3/10}|2\rangle + \sqrt{1/5}|6\rangle$ , then  $\mathcal{S}(\psi) = 3$  and  $\text{Spec}(\psi) = \{0, 2, 6\}$ .

Clearly, if  $\text{Spec}(\phi)$  is a rigid translation of  $\text{Spec}(\psi)$  then the transformation is possible. We write this sufficient condition as

$$\exists k \in \mathbb{Z} : \text{Spec}(\phi) = \text{Spec}(\psi) + k, \quad (39)$$

where  $\text{Spec}(\psi) + k \equiv \{n_0(\psi) + k, n_1(\psi) + k, \dots, n_{\mathcal{S}(\psi)}(\psi) + k\}$ . (One could also write the condition as  $\exists k \in \mathbb{Z} : \forall n \in \text{Spec}(\psi), n - k \in \text{Spec}(\phi)$ .) Note that  $k$  can be negative and consequently  $\text{Spec}(\psi) + k$  may have negative elements. However, if this occurs then  $\text{Spec}(\psi) + k$  cannot equal  $\text{Spec}(\phi)$  since the latter has only positive elements, and the  $k$  value in question is not one for which the transformation is possible.

Although the condition of Eq. (39) is sufficient, it is not necessary. Because a stochastic transformation can send a non-zero probability to zero,  $\text{Spec}(\phi)$  need only be a subset of a rigid translation of  $\text{Spec}(\psi)$ . Consequently, we have

**Theorem 4.** *The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible using stochastic  $U(1)$ -invariant operations if and only if*

$$\exists k \in \mathbb{Z} : \text{Spec}(\phi) \subset \text{Spec}(\psi) + k. \quad (40)$$

(One could also write the condition as  $\exists k \in \mathbb{Z} : \forall n \in \text{Spec}(\phi), n - k \in \text{Spec}(\psi)$ .) Here, we must include for consideration those  $k$  that yield negative elements for  $\text{Spec}(\psi) + k$  because these elements might be given zero amplitude by the operation.

An example is illustrated in Fig. 3. If  $\text{Spec}(\psi) = \{1, 3, 4, 6, 10, 11, 12\}$  and  $\text{Spec}(\phi) = \{7, 13, 14\}$ , the transformation is possible by sending to zero the weights of the number eigenstates  $|1\rangle, |3\rangle, |6\rangle$  and  $|12\rangle$ , and translating the number upward by  $k = 3$ , thereby transforming  $|4\rangle, |10\rangle$  and  $|11\rangle$  to  $|7\rangle, |13\rangle$  and  $|14\rangle$  respectively, and finally rescaling the weights to correspond to those of  $|\phi\rangle$ .

It is not difficult to see that the theorem must be true. Nevertheless, we provide an explicit proof.

**Proof.** Let  $|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle$  and  $|\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$ . Suppose Eq. (40) holds for some  $k$ , then we can achieve  $|\psi\rangle \rightarrow |\phi\rangle$  using the  $U(1)$ -invariant operation defined by the Kraus operator  $K_k = S_k \tilde{K}_k$  where  $\tilde{K}_k = \sum_n c_n^{(k)} |n\rangle \langle n|$  and the  $c_n^{(k)}$  are defined as follows: If  $n + k \in \text{Spec}(\phi)$ , then  $c_n^{(k)} \propto \sqrt{q_{n+k}}/\sqrt{p_n}$  (with norm chosen such that  $c_n^{(k)} \leq 1$ ), otherwise  $c_n^{(k)} = 0$ . Note that for  $n + k \in \text{Spec}(\phi)$ , we have  $q_{n+k} \neq 0$ . Also, given Eq. (40), if  $n + k \in \text{Spec}(\phi)$  then  $n \in \text{Spec}(\psi)$  and  $p_n \neq 0$ . Thus  $c_n^{(k)}$  is always well-defined. It is easily verified that by these definitions,  $K_k |\psi\rangle \propto |\phi\rangle$ .

Conversely, if  $K_k |\psi\rangle \propto |\phi\rangle$ , then supposing that  $\tilde{K}_k = \sum_n c_n^{(k)} |n\rangle \langle n|$ , we require  $(c_n^{(k)})^2 p_n \propto q_{n+k}$ . Eq. (40) follows. QED.

Note that there exist pairs of states,  $|\psi\rangle$  and  $|\phi\rangle$ , for which neither direction of transformation (neither  $|\psi\rangle \rightarrow |\phi\rangle$  nor  $|\phi\rangle \rightarrow |\psi\rangle$ ) is possible using stochastic  $U(1)$ -invariant operations. A simple example is the pair  $|0\rangle + |1\rangle$  and  $|0\rangle + |2\rangle$ .<sup>5</sup>

## 2. Maximum probability

Given two pure states  $|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle$  and  $|\phi\rangle = \sum_n \sqrt{q_n} |n\rangle$  such that the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible by stochastic  $U(1)$ -invariant operations, what is the maximum probability to convert  $|\psi\rangle$  into  $|\phi\rangle$ ? We have only been able to find the complete solution in a special case.

**Theorem 5.** *If there is only a single value of  $k$  such that the condition  $\text{Spec}(\phi) \subset \text{Spec}(\psi) + k$  holds, then the maximum probability of achieving the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  using  $U(1)$ -invariant operations is*

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \min_n \left( \frac{p_n}{q_{n+k}} \right).$$

**Proof.** Recall the proof of Thm. 3 where it was noted that for a  $U(1)$ -invariant operation with Kraus operators  $\{K_{k,\alpha}\}$  to achieve  $|\psi\rangle \rightarrow |\phi\rangle$  deterministically, it

must satisfy  $K_{k,\alpha} |\psi\rangle = \sqrt{w_{k,\alpha}} |\phi\rangle$  for all  $k$  and  $\alpha$ . To achieve the transformation stochastically, this condition need only hold for one or more pairs of values of  $k$  and  $\alpha$ . We can still deduce Eq. (38) for these pairs of values, which we denote by  $(k, \alpha) \in S$ , and it follows that we have  $w_{k,\alpha} = (c_n^{(k,\alpha)})^2 p_n / q_{n+k}$  for every  $n$ . The total probability of this transformation is therefore

$$w = \sum_{k,\alpha \in S} \frac{(c_n^{(k,\alpha)})^2 p_n}{q_{n+k}}.$$

The task is to maximize this quantity under variations of the  $c_n^{(k,\alpha)}$  subject to the constraint that  $\sum_{k,\alpha \in S} (c_n^{(k,\alpha)})^2 \leq 1$  for every  $n$ .

The assumption that  $|\psi\rangle \rightarrow |\phi\rangle$  can only be achieved for a single value of  $k$  implies that our sum may be restricted to this value,

$$w = \frac{\left( \sum_{\alpha \in S} (c_n^{(k,\alpha)})^2 \right) p_n}{q_{n+k}},$$

and given that  $\sum_{\alpha \in S} (c_n^{(k,\alpha)})^2 \leq 1$  for every  $n$ , we infer that  $w \leq \frac{p_n}{q_{n+k}}$  for every  $n$ . This set of inequalities is captured by the single inequality  $w \leq \min_n \left\{ \frac{p_n}{q_{n+k}} \right\}$ . By

choosing  $\sum_{\alpha \in S} (c_n^{(k,\alpha)})^2 = 1$  for the  $n$  that achieves the minimum, we can saturate the inequality. QED.

If  $\text{Spec}(\phi) \subset \text{Spec}(\psi) + k$  for several different values of  $k$ , then the optimization is much more difficult. It may be that there is an optimal  $k$  to use. Alternatively, it may be that the probabilities associated with different  $k$  values can be added because one can implement a measurement upon  $|\psi\rangle$  such that more than one outcome collapses the state to  $|\phi\rangle$ . This is what occurs deterministically with  $(|0\rangle + |1\rangle + |2\rangle + |3\rangle)/2 \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$ . As another example, if there are two values,  $k_1$  and  $k_2$ , that satisfy  $\text{Spec}(\phi) \subset \text{Spec}(\psi) + k$  and for which  $|k_1 - k_2| > n_{\mathcal{S}(\phi)}(\phi) - n_1(\phi)$  (so that  $\text{Spec}(\phi) - k_1$  and  $\text{Spec}(\phi) - k_2$  do not overlap) then the probability of the transformation is at least

$$w = \min_n \left( \frac{p_n}{q_{n+k_1}} \right) + \min_n \left( \frac{p_n}{q_{n+k_2}} \right).$$

The problem of finding the maximum probability in the general case remains open, although techniques analogous to those in Ref. [16] are likely to yield the solution.

## E. Stochastic $U(1)$ -frameness monotones

From Thm. (4), we see that the cardinality of the number spectrum is non-increasing under stochastic  $U(1)$ -invariant operations, that is,  $\mathcal{S}(\phi) \leq \mathcal{S}(\psi)$ . This cardinality therefore satisfies the definition of a stochastic frameness monotone. (Note that the amplitudes  $\sqrt{p_n} \equiv \langle n | \psi \rangle$  play an analogous role here to that of

<sup>5</sup> The distinction between  $|0\rangle + |1\rangle$  and  $|0\rangle + |2\rangle$  as quantum phase references is analogous to the distinction between  $(|0\rangle + |1\rangle)^{\otimes 2}$  and  $(|0\rangle + |2\rangle)^{\otimes 2}$  as *shared* quantum phase references which, as van Enk has noted [26], play distinct roles in the theory of static and dynamic quantum communication resources under a local  $U(1)$ -SSR.



the Schmidt coefficients in entanglement theory, and the number spectrum cardinality  $\mathcal{S}(\psi)$  is analogous to the Schmidt number.)

In fact, it is straightforward to see that there are other features of the number spectrum that define stochastic frameness monotones. Perhaps the most obvious such feature is the difference between the largest and the smallest element of the spectrum. But the nonincreasing property also holds true for the difference between the second-largest and the smallest element of the spectrum, the third-largest and the smallest, and so forth. Fig. 3 makes this feature evident.

We can thereby define stochastic frameness monotones in terms of these difference as

$$\begin{aligned}\mathcal{F}_1(\psi) &\equiv n_{\mathcal{S}(\psi)}(\psi) - n_1(\psi) \\ \mathcal{F}_2(\psi) &\equiv n_{\mathcal{S}(\psi)-1}(\psi) - n_1(\psi) \\ &\dots \\ \mathcal{F}_{\mathcal{S}(\psi)-1}(\psi) &\equiv n_2(\psi) - n_1(\psi)\end{aligned}$$

and

$$\mathcal{F}_j(\psi) \equiv 0 \text{ for } j \geq \mathcal{S}(\psi).$$

We have presented these in decreasing order,  $\mathcal{F}_{j+1}(\psi) < \mathcal{F}_j(\psi)$  for  $j < \mathcal{S}(\psi)$ .

That all stochastic frameness monotones should be nonincreasing is clearly a necessary condition for the possibility of a particular transformation. For instance,  $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  can not be converted to  $|\phi\rangle = (|0\rangle + |2\rangle)/\sqrt{2}$  even by stochastic  $U(1)$ -invariant operations because  $\mathcal{F}_1(\psi) < \mathcal{F}_1(\phi)$ <sup>6</sup>.

In order to characterize the necessary and sufficient conditions for stochastic interconversion in terms of these monotones, it is useful to define the set of nonzero monotones,

$$\text{Mons}(\psi) \equiv \{\mathcal{F}_1(\psi), \mathcal{F}_2(\psi), \dots, \mathcal{F}_{\mathcal{S}(\psi)-1}(\psi)\}.$$

We can easily infer from Eq. (40) the following alternative form of theorem 4.

**Proposition 6.** *The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible using stochastic  $U(1)$ -invariant operations if and only if*

$$\exists l \in \mathbb{N} : \text{Mons}(\phi) \subset \text{Mons}(\psi) - l \quad (41)$$

where  $\text{Mons}(\psi) - l \equiv \{\mathcal{F}_1(\psi) - l, \mathcal{F}_2(\psi) - l, \dots, \mathcal{F}_{\mathcal{S}(\psi)-1}(\psi) - l\}$ .

(One can also write this as  $\exists l \in \mathbb{N} : \forall k \in \text{Mons}(\phi), k + l \in \text{Mons}(\psi)$ ).

Returning to our previous example of  $\text{Spec}(\psi) = \{1, 3, 4, 6, 10, 11, 12\}$  and  $\text{Spec}(\phi) = \{7, 13, 14\}$ , we have

$\text{Mons}(\psi) = \{11, 10, 9, 5, 3, 2\}$  and  $\text{Mons}(\phi) = \{7, 6\}$ . Clearly,  $\text{Mons}(\psi) - 3 = \{8, 7, 6, 2, 0, -1\}$ , which includes  $\{7, 6\}$ , so the condition is satisfied for  $l = 3$ . Again, the figure makes this clear.

## F. Asymptotic transformations

In this section, we demonstrate the existence of reversible asymptotic transformations – and therefore the existence of a unique measure of  $U(1)$ -frameness – for pure states that have a *gapless* number spectrum. A gap occurs when there are values of  $n$  receiving zero probability between a pair of values of  $n$  receiving nonzero probability. For instance, the spectra  $\{2, 3, 4, 5\}$  and  $\{0, 1\}$  are gapless, while  $\{1, 2, 4, 5\}$  and  $\{1, 7, 9\}$  have gaps. (A gapless number spectrum can also be characterized as one that is uniform over its support, that is, for which  $n_{i+1}(\psi) = n_i(\psi) + 1$  for  $i = 1, \dots, \mathcal{S}(\psi) - 1$ .)

The unique measure is the scaled number variance

$$V(|\psi\rangle) \equiv 4 \left[ \langle \psi | \hat{N}^2 | \psi \rangle - \langle \psi | \hat{N} | \psi \rangle^2 \right], \quad (42)$$

where the normalization is chosen in such a way that the state  $(|0\rangle + |1\rangle)/\sqrt{2}$  has unit variance.

**Theorem 7.** *The unique asymptotic measure of  $U(1)$ -frameness for pure states  $|\psi\rangle$  that have gapless number spectra is the variance,*

$$F^\infty(|\psi\rangle) = V(|\psi\rangle).$$

Given the choice of normalization, it follows that  $V(|\psi\rangle)$  quantifies the rate at which one can distill copies of  $(|0\rangle + |1\rangle)/\sqrt{2}$ , which may be considered to be one “bit” of phase reference. van Enk [26] has introduced the term *refbit* for the bipartite state  $(|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$ , which can be considered to be one “bit” of a *shared* phase reference. We suggest that it may be judicious to call the latter a *shared refbit*, while the state  $(|0\rangle + |1\rangle)/\sqrt{2}$  is called a *local refbit*.

This theorem is the adaptation to the unipartite context of the main result from Ref. [9] (where the measure was called the *superselection-induced variance*). Although the proof can be easily inferred from its bipartite counterpart in Ref. [9], for the sake of completeness and pedagogy, at the end of this section we provide a proof that is native to the unipartite context.

Note that the variance is not only weakly additive (as it must be given the discussion in Sec. II F), but strongly additive as well; that is, given two finite dimensional pure states  $|\psi\rangle$  and  $|\phi\rangle$  we have

$$V(|\psi\rangle \otimes |\phi\rangle) = V(|\psi\rangle) + V(|\phi\rangle).$$

Finally, we note that not only is the variance a deterministic monotone (as it must be given the discussion in Sec. II F), it is an ensemble monotone as well.

**Lemma 8.**  *$V(|\psi\rangle)$  is an ensemble frameness monotone.*

<sup>6</sup> However, as noted by van Enk [26], two copies of  $|0\rangle + |1\rangle$  can be converted with some probability to a single copy of  $|0\rangle + |2\rangle$ .

**Proof.** Under the U(1)-SSR, a transition from a state to an ensemble is induced by a U(1)-invariant measurement, that is, a measurement for which each outcome is associated with a U(1)-invariant operation. Suppose the outcome  $\mu$  occurs with probability  $w_\mu$  and is associated with a U(1)-invariant operation with Kraus decomposition  $\{K_{k,\alpha}^{(\mu)}|k, \alpha\}$  of the form specified in Lemma 2.

Given that each outcome leaves the system in a fixed state  $|\phi_\mu\rangle$ , we have that

$$K_{k,\alpha}^{(\mu)}|\psi\rangle = \sqrt{w_{\mu,k,\alpha}}|\phi_\mu\rangle$$

for all  $k$  and  $\alpha$ , where

$$w_\mu = \sum_{k,\alpha} w_{\mu,k,\alpha}. \quad (43)$$

The average value of  $V$  in the final ensemble is  $\sum_\mu w_\mu V(|\phi_\mu\rangle)$ .

Now note that there is a fine-graining of this measurement where each outcome is associated with the U(1)-invariant operation that has the single Kraus operator  $K_{k,\alpha}^{(\mu)}$ , so that the outcomes are labeled not only by  $\mu$ , but by  $k$  and  $\alpha$  as well and each has probability  $w_{\mu,k,\alpha}$  of occurring. The average value of  $V$  for the ensemble generated by this measurement is  $\sum_{\mu,k,\alpha} w_{\mu,k,\alpha} V(|\phi_\mu\rangle)$ , but because  $|\phi_\mu\rangle$  does not depend on  $k$  and  $\alpha$ , Eq. (43) implies that the average value of  $V$  is the same as for the original measurement. It suffices therefore to show that  $V$  is an ensemble monotone for the fine-grained measurement.

We redefine  $\mu$  to run over the outcomes of this fine-grained measurement. Each outcome is associated with a Kraus operation  $K_\mu$  which, by lemma 2, has the form

$$K_\mu = \sum_n c_n^{(\mu)} |n + k_\mu\rangle \langle n|,$$

where the  $c_n^{(\mu)}$  are complex coefficients and the  $k_\mu$  are integers. We therefore have

$$[\hat{N}, K_\mu] = k_\mu K_\mu \text{ and } [\hat{N}^2, K_\mu] = 2k_\mu K_\mu \hat{N} + k_\mu^2 K_\mu$$

Now, after an outcome  $\mu$  has occurred, the state of the system is  $|\phi_\mu\rangle = \frac{1}{\sqrt{w_\mu}} K_\mu |\psi\rangle$ , where  $w_\mu = \langle \psi | K_\mu^\dagger K_\mu | \psi \rangle$ .

Thus, on average

$$\begin{aligned} & \sum_\mu w_\mu V(|\phi_\mu\rangle) \\ &= 4 \sum_\mu \left( \langle \psi | K_\mu^\dagger \hat{N}^2 K_\mu | \psi \rangle - \frac{\langle \psi | K_\mu^\dagger \hat{N} K_\mu | \psi \rangle^2}{w_\mu} \right) \end{aligned}$$

From the commutation relations above we conclude that:

$$\begin{aligned} \sum_\mu \langle \psi | K_\mu^\dagger \hat{N}^2 K_\mu | \psi \rangle &= \langle \psi | \hat{N}^2 | \psi \rangle \\ &+ 2 \sum_\mu k_\mu \langle \psi | K_\mu^\dagger K_\mu \hat{N} | \psi \rangle + \sum_\mu w_\mu k_\mu^2, \end{aligned}$$

and

$$\begin{aligned} \sum_\mu \frac{\langle \psi | K_\mu^\dagger \hat{N} K_\mu | \psi \rangle^2}{w_\mu} &= \sum_\mu \frac{1}{w_\mu} \langle \psi | K_\mu^\dagger K_\mu \hat{N} | \psi \rangle^2 \\ &+ 2 \sum_\mu k_\mu \langle \psi | K_\mu^\dagger K_\mu \hat{N} | \psi \rangle + \sum_\mu w_\mu k_\mu^2 \end{aligned}$$

where for the upper equation we have used the fact that  $\sum_\mu K_\mu^\dagger K_\mu = I$ . We therefore obtain

$$\sum_\mu w_\mu V(|\phi_\mu\rangle) = 4 \left( \langle \psi | \hat{N}^2 | \psi \rangle - \sum_\mu \frac{\langle \psi | K_\mu^\dagger K_\mu \hat{N} | \psi \rangle^2}{w_\mu} \right).$$

Now, let  $x_\mu \equiv \langle \psi | K_\mu^\dagger K_\mu \hat{N} | \psi \rangle$ . From the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_\mu \frac{x_\mu^2}{w_\mu} &= \sum_\mu \frac{x_\mu^2}{w_\mu} \sum_{\mu'} w_{\mu'} \geq \left( \sum_\mu \frac{x_\mu}{\sqrt{w_\mu}} \sqrt{w_\mu} \right)^2 \\ &= \left( \sum_\mu x_\mu \right)^2 = \langle \psi | \hat{N} | \psi \rangle^2, \end{aligned}$$

where for the last equality we have again used the fact that  $\sum_\mu K_\mu^\dagger K_\mu = I$ . We therefore have

$$\sum_\mu w_\mu V(|\phi_\mu\rangle) \leq 4 \left( \langle \psi | \hat{N}^2 | \psi \rangle - \langle \psi | \hat{N} | \psi \rangle^2 = V(|\psi\rangle) \right).$$

Thus, the variance of  $\hat{N}$  is non-increasing on average under U(1)-invariant operations.

Note that this proof follows that of Schuch *et al.* [10] for the ensemble monotonicity of the superselection-induced variance but generalizes the latter insofar as it incorporates the possibility of shifts in the value of  $n$ . QED.

**Proof of theorem 7.** Suppose the state is given in the standard form  $|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle$ . The assumption that  $|\psi\rangle$  has a gapless number spectrum implies that  $p_n \neq 0$  for all  $n$  in the range  $n_1(\psi)$  to  $n_{\mathcal{S}(\psi)}(\psi)$ . However, we can shift the number downward by  $n_1(\psi)$  using a U(1)-invariant operation to obtain

$$|\psi\rangle = \sum_{n=0}^d \sqrt{\tilde{p}_n} |n\rangle,$$

where  $d \equiv \mathcal{S}(\psi) - 1$  and  $\tilde{p}_n = p_{n+n_1(\psi)}$ . We therefore assume a state of this form in what follows.

We would like to write an expression for  $|\psi\rangle^{\otimes N}$  in the standard form. Recall that all the terms in the resulting expression with the same total number eigenvalue can be transformed, via a U(1)-invariant operation to a single term which we denote by  $|n\rangle$  (for instance,  $|0\rangle|2\rangle|2\rangle$ ,  $|1\rangle|1\rangle|2\rangle$ , and  $|1\rangle|0\rangle|3\rangle$  can all be transformed to  $|4\rangle$ ). Keeping this in mind and using the multinomial formula, we have

$$|\psi\rangle^{\otimes N} = \sum_{n=0}^{dN} \sqrt{r_n} |n\rangle, \quad (44)$$

with

$$r_n \equiv \sum \frac{n!}{N_1! N_2! \dots N_{d-1}!} \tilde{p}_0^{N_0} \tilde{p}_1^{N_1} \dots \tilde{p}_{d-1}^{N_{d-1}},$$

where the sum is taken over all nonnegative integers  $N_0, N_1, \dots, N_d$  for which  $\sum_{n'=0}^d N_{n'} = N$  and  $\sum_{n'=0}^d n' N_{n'} = n$ . In the limit  $N \rightarrow \infty$ , the distribution  $r_n$  approaches a Gaussian as long as for all  $n \in \{0, \dots, d\}$ , we have  $\tilde{p}_n > 0$  [10]. The proof is blocked if  $\tilde{p}_n = 0$  for some  $n$  in this range and it is for this reason that our theorem is restricted to pure states with gapless number spectra.

A Gaussian distribution depends only on two parameters: the mean and the variance. However, in the limit of large  $N$ , the mean can be shifted freely by  $U(1)$ -invariant operations. This follows from lemma 2 and the fact that the amplitude of the Gaussian at  $n = 0$  is negligible in the limit of large  $N$ . Hence, in the limit  $N \rightarrow \infty$ , there exists an allowed  $\mathcal{E}$  such that  $\text{Fid}(\mathcal{E}(|\psi\rangle^{\otimes N}), |\phi\rangle^{\otimes M}) \rightarrow 1$  as long as  $|\psi\rangle^{\otimes N}$  and  $|\phi\rangle^{\otimes M}$  have the same variance,  $V(|\psi\rangle^{\otimes N}) = V(|\phi\rangle^{\otimes M})$ . Given the additivity of the variance, we conclude that in the limit  $N \rightarrow \infty$ ,  $MV(|\phi\rangle) = NV(|\psi\rangle)$ , which is the result we sought to demonstrate.

Note that although the only maps that are reversible on *all* states are unitary maps, in the present context reversibility of the maps is only required on states of the form  $|\psi\rangle^{\otimes N}$ , a constraint that can be met by nonunitary operations such as those induced by the shift operator  $S_k$ . This fact is critical in the present context because the mean can only be changed by such operations. Of course, the fact that the change must be accomplished only imperfectly is critical here. QED.

Finding the asymptotic rate of interconversion between a pair of states  $|\psi\rangle$  and  $|\phi\rangle$ , one or both of which have gapped spectra, remains an open problem. Nonetheless, we present a few observations on the general problem here.

First, it is clear that the asymptotic rate of interconversion can be strictly zero. The case  $|\phi\rangle \rightarrow |\psi\rangle$  where  $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  (with a gapless spectrum) and  $|\phi\rangle = (|0\rangle + |2\rangle)/\sqrt{2}$  (with a gapped spectrum) provide the simplest example. The rate is zero because  $|\phi\rangle^{\otimes M}$  has no weight on odd numbers for any  $M$ , while  $|\psi\rangle^{\otimes N}$  does for all  $N$ . (Schuch *et al.* [10] point out that small amounts of additional resources can catalyze the asymptotic interconversion, but strictly speaking the rate is zero.)

Second, the rate of interconversion can be zero in one direction but nonzero in the other. Indeed, the states  $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  and  $|\phi\rangle = (|0\rangle + |2\rangle)/\sqrt{2}$  are such a case. In the limit  $N \rightarrow \infty$ , the weights  $r_n = |\langle n | |\psi\rangle^{\otimes N} \rangle|^2$  form a Gaussian with variance  $NV(|\psi\rangle)$ , while the weights  $s_n = |\langle n | |\phi\rangle^{\otimes M} \rangle|^2$  are zero for all odd values of  $n$  but lie under a Gaussian envelope with variance  $MV(|\phi\rangle)$ . Recalling Thm. 3,  $|\psi\rangle^{\otimes N}$  can be transformed deterministically to  $|\phi\rangle^{\otimes M}$  if and only if  $\vec{r} = \sum_k w_k \Upsilon_k \vec{s}$ . This condition is indeed satisfied (at

least approximately) when the variances of  $\vec{r}$  and  $\vec{s}$  are equal, i.e. when  $\lim_{N \rightarrow \infty} M/N = V(|\psi\rangle)/V(|\phi\rangle)$ , because in this case  $\vec{r} \simeq (1/2)\vec{s} + (1/2)\Upsilon_1 \vec{s}$ . The deterministic transformation is achieved by measuring whether  $n$  is even or odd, and upon finding it odd, shifting its value upward by  $k = 1$ . More precisely, the  $U(1)$ -invariant operation that achieves  $|\psi\rangle^{\otimes N} \rightarrow |\phi\rangle^{\otimes M}$  is the one that has Kraus operators  $K_0 = \sum_{n \text{ even}} |n\rangle \langle n|$  and  $K_1 = S_1 \sum_{n \text{ odd}} |n\rangle \langle n|$ . As noted above, the opposite transformation,  $|\phi\rangle^{\otimes N} \rightarrow |\psi\rangle^{\otimes M}$ , cannot be achieved with any rate. It is useful to justify this in terms of the condition for deterministic transformations. The condition requires that  $\vec{s} = \sum_k w_k \Upsilon_k \vec{r}$  but this cannot be satisfied (even approximately) as no convex combination of shifted versions of a gapless spectrum can yield a gapped spectrum. In brief, under a deterministic  $U(1)$ -invariant operation, gaps can be created but they cannot be filled. We see once again that there are distinct inequivalent sorts of resources under the  $U(1)$ -SSR.

Third, the rate of interconversion is not a continuous function. Consider the unnormalized and gapless states  $|0\rangle + \epsilon|1\rangle + |2\rangle$  and  $|0\rangle + |1\rangle + \epsilon|2\rangle$ , where  $\epsilon$  is a positive real small number. Because the variance of the first state is greater than that of the second, the rate of converting the first to the second is greater than 1 for sufficiently small epsilon. However, the rate must jump discontinuously to zero if we take  $\epsilon$  to zero.

Finally, note that the class of states with gapless number spectra is not the only class for which reversible asymptotic transformations exist. Many other examples can be found, such as the class of states with gaps of width  $x$  for some fixed  $x > 0$ .

Clearly, there remains much work to be done to completely characterize asymptotic transformations under a  $U(1)$ -SSR for states with arbitrary number spectra.

## IV. RESOURCE THEORY OF THE $Z_2$ -SSR

### A. Chiral frames

The second example we consider is that of a reference frame for chirality. Such a frame is the component of a Cartesian frame with respect to which the handedness of a quantum system is defined. The space inversion  $\vec{x} \rightarrow -\vec{x}$  is the coordinate transformation that changes a right-handed system into a left-handed one and vice-versa. Performing space inversion twice is equivalent to performing the identity transformation  $\vec{x} \rightarrow \vec{x}$ . These two transformations are a representation of the group  $Z_2$ . We label the two elements of  $Z_2$  by  $e$  and  $f$  (identity and flip). Their representation on Hilbert space is

$$\begin{aligned} T(e) &= I, \\ T(f) &= \pi, \end{aligned}$$

where  $\pi$  is the parity operator. Because the parity operator is Hermitian and satisfies  $\pi^2 = I$ , its eigenvalues are  $\pm 1$ .

For a single quantum particle, the action of the parity operator is

$$\pi |l, m\rangle = (-1)^l |l, m\rangle, \quad (45)$$

where  $l$  and  $m$  are the orbital angular momentum quantum numbers –  $l(l+1)$  is the eigenvalue of  $\mathbf{L}^2$  and  $\hbar m$  is the eigenvalue of  $\mathbf{L}_z$ . The parity is even (eigenvalue +1) for  $l$  even and it is odd (eigenvalue -1) for  $l$  odd. Equation (45) is easily verified by noting that  $\langle \vec{x} | (\pi |l, m\rangle) = \langle -\vec{x} | l, m\rangle = Y_l^m(-\vec{x}) = (-1)^l Y_l^m(\vec{x}) = (-1)^l \langle \vec{x} | l, m\rangle$ , where the  $Y_l^m(\vec{x})$  denote the spherical harmonics, and where we have made use of the fact that these are eigenfunctions of the space inversion operation with eigenvalue  $(-1)^l$  (see e.g. p. 255 in Sakurai [17]).

For  $N$  quantum particles, the representation of space inversion is simply the tensor product representation,  $\pi \equiv T(f) = \bigotimes_{i=1}^N T_i(f) = \bigotimes_{i=1}^N \pi_i$ , where  $\pi_i$  is the parity operator for the  $i$ th particle. This acts as

$$\pi \bigotimes_i |l_i, m_i\rangle = (-1)^{\sum_i l_i} \bigotimes_i |l_i, m_i\rangle.$$

Consequently the collective parity is even (odd) if the sum  $\sum_i l_i$  of the orbital angular momentum quantum numbers of the components is even (odd).

Note that a spin system is invariant under space inversion. It follows that no state of a spin system can act as a quantum reference frame for chirality. Only quantum particles can constitute such a resource (see e.g. p. 254 in [17]).

The states of the quantum particle that are invariant under space inversion (i.e. the  $Z_2$ -invariant states) are those satisfying

$$\pi \rho \pi = \rho,$$

or equivalently,

$$[\rho, \pi] = 0.$$

Again, it is useful to decompose the Hilbert space into the eigenspaces of even and odd parity (the carrier spaces for the irreducible representations of  $Z_2$ ),

$$\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}}.$$

Let  $\{|b, \beta\rangle\}$  be a basis for  $\mathcal{H}$ , where  $b$  is a bit specifying the parity and  $\beta$  is a multiplicity index,

$$\pi |b, \beta\rangle = (-1)^b |b, \beta\rangle.$$

Clearly,  $\{|0, \beta\rangle\}$  is a basis for  $\mathcal{H}_{\text{even}}$  and  $\{|1, \beta\rangle\}$  is a basis for  $\mathcal{H}_{\text{odd}}$ .

For instance, for a single quantum particle, we can take

$$|b, \beta\rangle = |l, m\rangle,$$

where

$$b = l \bmod 2 = \begin{cases} 0 & \text{if } l \text{ is even} \\ 1 & \text{if } l \text{ is odd} \end{cases}$$

and  $\beta$  is an index that specifies the remaining information in  $(l, m)$ , specifically,

$$\beta = \left(\frac{l-b}{2}, m\right).$$

Similarly, for  $N$  quantum particles, we can take

$$|b, \beta\rangle = \bigotimes_{i=1}^N |l_i, m_i\rangle,$$

where

$$b = \left(\sum_i l_i\right) \bmod 2 = \begin{cases} 0 & \text{if } \sum_i l_i \text{ is even} \\ 1 & \text{if } \sum_i l_i \text{ is odd} \end{cases}$$

and  $\beta$  is an index that specifies the remaining information in  $l_1, m_1, l_2, m_2, \dots$ .

It is clear that any change to the multiplicity index does not require a reference frame for chirality. It follows that any operation within  $\mathcal{H}_{\text{even}}$  or within  $\mathcal{H}_{\text{odd}}$  is possible under the  $Z_2$ -SSR and that any pure state  $|\psi\rangle = \sum_{b, \beta} \lambda_{b, \beta} |b, \beta\rangle$  can be taken, by a  $Z_2$ -invariant unitary operator, to the form

$$|\psi\rangle = \sum_b \lambda_b |b\rangle = \lambda_0 |0\rangle + \lambda_1 |1\rangle,$$

where  $|0\rangle$  and  $|1\rangle$  are a pair of standard states of even and odd parity respectively,  $\pi |b\rangle = (-1)^b |b\rangle$ . In what follows, we will assume that all pure states have been transformed into this standard form. We are therefore working in the two-dimensional subspace  $\mathcal{H}' = \text{span}(|0\rangle, |1\rangle)$ .

Note that only the states  $|0\rangle$  and  $|1\rangle$  are invariant under space inversion. Any coherent superposition of these is therefore a resource under the  $Z_2$ -SSR. Such quantum states, which act as bounded-size reference frames for chirality, have been dubbed “quantum gloves” in recent work [24, 25].

The physical significance of the  $Z_2$ -SSR is clarified by considering a scenario wherein two parties, Alice and Charlie, fail to share a reference frame for chirality. If the state of a system is  $\rho$  relative to Alice’s frame, then relative to Charlie’s frame the state is described by the  $Z_2$ -twirling of  $\rho$ ,

$$\mathcal{Z}[\rho] \equiv \frac{1}{2}\rho + \frac{1}{2}\pi\rho\pi.$$

An eigenstate of parity relative to Alice’s frame appears as the same state relative to Charlie’s frame. On the other hand, a superposition of such states is a resource in the sense that it provides for Charlie a token of Alice’s chiral frame, one which Charlie cannot prepare himself.

It should be noted that there are many other restrictions on operations, besides the lack of a chiral reference frame, that are described by a  $Z_2$ -SSR. For instance, having a reference frame for phase modulo  $\pi$  is a restriction relative to possessing a full phase reference and is described by a  $Z_2$ -SSR. It may arise, for instance, if Alice

and Bob have local phase references and are uncertain of whether they are in phase or  $\pi$  out of phase. This is clearly a milder restriction than knowing nothing about the relative phase. We highlight this example because it provides a physical explanation of why the  $U(1)$ -SSR is a stronger restriction than the  $Z_2$ -SSR. Nonetheless, when we attempt to characterize our results in physical terms, we shall use the lack of a chiral reference frame as our example.

### B. $Z_2$ -invariant operations

We now turn to the  $Z_2$ -invariant operations. Lemma 1 provides a characterization. First note that  $Z_2$  has only two irreducible representations, both 1-dimensional, which we label by  $B \in \{0, 1\}$  and denote by  $u_B : Z_2 \rightarrow \mathbb{C}$ . Denoting the elements of  $Z_2$  by  $e$  and  $f$ , the irreps are

$$\begin{aligned} u_0(e) &= 1, \quad u_0(f) = 1, \text{ and} \\ u_1(e) &= 1, \quad u_1(f) = -1. \end{aligned}$$

It follows from Lemma 1 that a  $Z_2$ -invariant operation has Kraus operators  $K_{B,\alpha}$ , labelled by an irrep  $B$  and a multiplicity index  $\alpha$ , satisfying

$$\begin{aligned} \pi K_{0,\alpha} \pi &= K_{0,\alpha}, \\ \pi K_{1,\alpha} \pi &= -K_{1,\alpha}. \end{aligned}$$

Just as in the  $U(1)$  case, the fact that the irreps are 1-dimensional implies that the action of  $Z_2$  does not mix these Kraus operators, which simplifies the analysis.

Confining ourselves to the two-dimensional subspace  $\mathcal{H}' = \text{span}(|0\rangle, |1\rangle)$ , we infer that  $K_{0,\alpha}$  is a  $Z_2$ -invariant operator of the form

$$\begin{aligned} K_{0,\alpha} &= a_\alpha |0\rangle\langle 0| + b_\alpha |1\rangle\langle 1|, \\ &= \begin{pmatrix} a_\alpha & 0 \\ 0 & b_\alpha \end{pmatrix}, \end{aligned} \quad (46)$$

while  $K_{1,\alpha}$  has the form

$$\begin{aligned} K_{1,\alpha} &= c_\alpha |1\rangle\langle 0| + d_\alpha |0\rangle\langle 1|, \\ &= \begin{pmatrix} 0 & d_\alpha \\ c_\alpha & 0 \end{pmatrix}. \end{aligned} \quad (47)$$

In order for the operation to be trace-nonincreasing, we require  $\sum_{B,\alpha} K_{B,\alpha}^\dagger K_{B,\alpha} \leq I$ , which implies that  $\sum_\alpha (|a_\alpha|^2 + |c_\alpha|^2) \leq 1$  and  $\sum_\alpha (|b_\alpha|^2 + |d_\alpha|^2) \leq 1$ , where the inequalities are saturated if the operation is trace-preserving.

We summarize this result by the following lemma:

**Lemma 9.** *A  $Z_2$ -invariant operation admits a Kraus decomposition  $\{K_{B,\alpha}\}$ , where  $B \in \{0, 1\}$  and  $\alpha$  is an integer, satisfying*

$$K_{B,\alpha} = S_B \tilde{K}_{B,\alpha} \quad (48)$$

where

$$\begin{aligned} \tilde{K}_{B,\alpha} &\equiv c_0^{(B,\alpha)} |0\rangle\langle 0| + c_1^{(B,\alpha)} |1\rangle\langle 1| \\ &= \begin{pmatrix} c_0^{(B,\alpha)} & 0 \\ 0 & c_1^{(B,\alpha)} \end{pmatrix} \end{aligned}$$

changes the relative amplitudes of the parity eigenstates, and

$$\begin{aligned} S_0 &\equiv I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S_1 &\equiv X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

are, respectively, the identity operator, which leaves the parity unchanged, and the Pauli  $X$  operator, which flips the parity. The coefficients satisfy  $\sum_{B,\alpha} |c_b^{(B,\alpha)}|^2 \leq 1$  for all  $b$ , with equality if the operation is trace-preserving.

This Kraus decomposition is analogous to the one specified in Lemma 2 for  $U(1)$ -invariant operations.

#### 1. $Z_2$ -invariant unitaries

We have already mentioned how all unitaries that act within the spaces  $\mathcal{H}_{\text{even}}$  and  $\mathcal{H}_{\text{odd}}$  are  $Z_2$ -invariant. Indeed, it is because of this fact that every state can be transformed to one of the form  $|\psi\rangle = \lambda_0 |0\rangle + \lambda_1 |1\rangle$ . There are, however, additional  $Z_2$ -invariant unitaries. Consider the two sorts of irreducible operations described in the previous section. For an operation to be unitary, the associated Kraus operator must be a unitary operator.

For  $K_{0,\alpha}$  to be unitary, we require that  $|a|^2 = |b|^2 = 1$  in Eq. (46). Such an operation can still change the relative phase of  $|0\rangle$  and  $|1\rangle$ . It follows that any state  $|\psi\rangle = \lambda_0 |0\rangle + \lambda_1 |1\rangle$  can be transformed by a  $Z_2$ -invariant unitary into one of the form

$$|\psi\rangle = \sqrt{p_0} |0\rangle + \sqrt{p_1} |1\rangle,$$

where  $p_0 + p_1 = 1$ , which is to say, a form with real-amplitude coefficients.

In addition, the bit flip operation  $X$  is unitary, which implies that any state can be transformed to one of the form

$$|\psi\rangle = \sqrt{p_0} |0\rangle + \sqrt{p_1} |1\rangle, \text{ where } p_0 \geq p_1.$$

We will use both of these forms in what follows. If the form with ordered weights is being used, then this assumption will be made explicit.

### C. Deterministic single-copy transformations

We wish to determine when the transformations  $|\psi\rangle \rightarrow |\phi\rangle$  can be achieved *deterministically* using only  $Z_2$ -invariant operations.

We begin by defining a measure of  $Z_2$ -frameness that will be significant in what follows.

**Definition:** For a state of the form  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$ , we define the measure  $\mathcal{C}$  by

$$\mathcal{C}(|\psi\rangle) \equiv 2\min\{p_0, p_1\} \quad (49)$$

Note that if the state is written in the standard form where  $p_0 \geq p_1$  then the measure is simply expressed as

$$\mathcal{C}(|\psi\rangle) \equiv 2p_1.$$

We choose a normalization factor of 2 so that  $0 \leq \mathcal{C}(|\psi\rangle) \leq 1$ .

As we will see in Sec. IVE, this measure has an operational interpretation:  $\mathcal{C}(\psi)/\mathcal{C}(\phi)$  determines the maximum probability to convert  $\psi$  into  $\phi$  using only  $Z_2$ -invariant operations. Also, in Sec. IVD we show that it is an ensemble monotone and satisfies several interesting properties. This measure also helps us to express the criterion for deterministic single-copy transformations.

**Theorem 10.** *The necessary and sufficient condition for the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  to be possible by a deterministic  $Z_2$ -invariant operation is*

$$\mathcal{C}(|\psi\rangle) \geq \mathcal{C}(|\phi\rangle). \quad (50)$$

Note that if we take  $|\psi\rangle$  and  $|\phi\rangle$  to be in the standard forms,

$$\begin{aligned} |\psi\rangle &= \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle \text{ where } p_0 \geq p_1 \\ |\phi\rangle &= \sqrt{q_0}|0\rangle + \sqrt{q_1}|1\rangle \text{ where } q_0 \geq q_1. \end{aligned}$$

then the condition can be expressed as

$$p_0 \leq q_0, \quad (51)$$

which is equivalent to

$$\vec{p} \text{ is majorized by } \vec{q} \quad (52)$$

where the notion of majorization is defined in Ref. [34].

**Proof.** Recall from lemma 9 that a general  $Z_2$ -invariant operation has a Kraus decomposition  $\{K_{B,\alpha}\}$  where

$$K_{B,\alpha} = S_B \begin{pmatrix} c_0^{(B,\alpha)} & 0 \\ 0 & c_1^{(B,\alpha)} \end{pmatrix}.$$

For a deterministic transformation  $|\psi\rangle \rightarrow |\phi\rangle$ , we require

$$K_{B,\alpha}|\psi\rangle = \sqrt{w_{B,\alpha}}|\phi\rangle, \quad (53)$$

for all  $B, \alpha$  where  $0 \leq w_{B,\alpha} \leq 1$  and  $\sum_{B,\alpha} w_{B,\alpha} = 1$ . The case  $B = 0$  yields

$$c_0^{(B,\alpha)}\sqrt{p_0} = \sqrt{w_{B,\alpha}}\sqrt{q_0},$$

whereas the case  $B = 1$  yields

$$c_1^{(B,\alpha)}\sqrt{p_1} = \sqrt{w_{B,\alpha}}\sqrt{q_1}$$

Therefore,

$$K_{0,\alpha} = \sqrt{w_{0,\alpha}} \begin{pmatrix} \sqrt{q_0/p_0} & 0 \\ 0 & \sqrt{q_1/p_1} \end{pmatrix} \quad (54)$$

$$K_{1,\alpha} = X\sqrt{w_{1,\alpha}} \begin{pmatrix} \sqrt{q_1/p_0} & 0 \\ 0 & \sqrt{q_0/p_1} \end{pmatrix}. \quad (55)$$

For the transformation to be deterministic, we require that  $\sum_{B,\alpha} \langle\psi|K_{B,\alpha}^\dagger K_{B,\alpha}|\psi\rangle = 1$ , which implies that  $\sum_b \left[ \sum_{B,\alpha} (c_b^{(B,\alpha)})^2 \right] p_b = 1$ , and consequently that  $\sum_{B,\alpha} (c_b^{(B,\alpha)})^2 = 1$  for  $b = 0$  and  $b = 1$ . It follows that

$$\begin{aligned} w_0 q_0 + w_1 q_1 &= p_0, \\ w_0 q_1 + w_1 q_0 &= p_1. \end{aligned}$$

where

$$w_B \equiv \sum_{\alpha} w_{B,\alpha},$$

so that  $0 \leq w_B \leq 1$  and  $\sum_B w_B = 1$ . Solving these for  $w_0$  and  $w_1$ , we have

$$w_0 = \frac{p_0 q_0 - p_1 q_1}{q_0^2 - q_1^2} \quad (56)$$

$$w_1 = \frac{p_0 q_1 - p_1 q_0}{q_1^2 - q_0^2}. \quad (57)$$

Recalling that  $q_0 > q_1$  and  $p_0 > p_1$ , these two conditions imply that  $q_0 \geq p_0$ .

Conversely, if  $q_0 \geq p_0$ , then the operation defined by the pair of Kraus operators of Eqs. (54) and (55) (with no degeneracy index  $\alpha$ ) and with  $w_0$  and  $w_1$  given by Eqs. (56) and (57) is a  $Z_2$ -invariant trace-preserving operation that achieves the transformation  $|\psi\rangle \rightarrow |\phi\rangle$ . QED.

Consider now the state  $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$  with degenerate weights,  $p_0 = p_1 = \frac{1}{2}$ . It is a maximal resource in the sense that it can be deterministically transformed into any other state with nondegenerate weights. The reason is that for all states of the standard form  $|\phi\rangle = \sqrt{q_0}|0\rangle + \sqrt{q_1}|1\rangle$  where  $q_0 \geq q_1$ , we have  $q_0 \geq 1/2$ , which implies that the condition of Eq. (51) is satisfied. Indeed, the Kraus operators of the operation that implements the transformation  $|+\rangle \rightarrow |\phi\rangle$  are simply  $K_0 = \text{diag}(\sqrt{2q_0}, \sqrt{2q_1})$  and  $K_1 = X \text{diag}(\sqrt{2q_1}, \sqrt{2q_0})$ . Conversely, any state of the standard form  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$  where  $p_0 > p_1$ , i.e. with nondegenerate weights, cannot be deterministically transformed into  $|+\rangle$  because  $p_0 > 1/2$  and Eq. (51) fails to be satisfied.

## D. Ensemble $Z_2$ -frameness monotones

It is shown here that not only is  $\mathcal{C}$  an ensemble  $Z_2$ -frameness monotone, but every such monotone is a non-decreasing concave function of  $\mathcal{C}$ .

**Lemma 11.**  $\mathcal{C}(|\psi\rangle)$  is an ensemble  $Z_2$ -frameness monotone.

**Proof.** A transition from a state to an ensemble occurs as the result of a  $Z_2$ -invariant measurement, that is, a measurement for which each outcome is associated with a  $Z_2$ -invariant operation. For the same reasons provided in the proof of lemma 8, it suffices to consider measurements for which each outcome is associated with an operation with a single Kraus operator (all other measurements can be obtained by coarse-graining of these and this process does not change the value of the monotone).

Suppose the outcome  $\mu$  occurs with probability  $w_\mu$  and is associated with a Kraus operator  $K_\mu$  which, by lemma 9, has the form

$$K_\mu = S_{B_\mu} \begin{pmatrix} c_0^{(\mu)} & 0 \\ 0 & c_1^{(\mu)} \end{pmatrix},$$

where  $\sum_\mu |c_b^{(\mu)}|^2 \leq 1$  for  $b = 0$  and  $1$ , and where  $B_\mu$  is a bit, with  $S_0 = I$  while  $S_1 = X$ . We then have

$$\begin{aligned} \sum_\mu w_\mu \mathcal{C}(|\phi_\mu\rangle) &= \sum_\mu \mathcal{C}(K_\mu |\psi\rangle) \\ &= \sum_\mu \min\{|c_0^{(\mu)}|^2 p_0, |c_1^{(\mu)}|^2 p_1\} \\ &\leq \min\{p_0, p_1\} = \mathcal{C}(|\psi\rangle). \end{aligned}$$

QED.

Every non-decreasing concave function of  $\mathcal{C}$  is also an ensemble monotone, as noted in Sec. II C. (Recall that  $f$  is concave if  $f(wx + (1-w)y) \geq wf(x) + (1-w)f(y)$  for all  $w, x, y \in [0, 1]$ .) What is particularly interesting about  $\mathcal{C}$  however is that the opposite implication also holds true.

**Theorem 12.** Every ensemble  $Z_2$ -frameness monotone is a non-decreasing concave function of  $\mathcal{C}$ .

The proof of this theorem makes use of the following theorem concerning the transformation  $|\psi\rangle \rightarrow \{(w_\mu, |\varphi_\mu\rangle)\}$  of a pure state  $|\psi\rangle$  to an ensemble  $\{(w_\mu, |\varphi_\mu\rangle)\}$ . Such a transformation is achieved if there is a measurement that collapses  $|\psi\rangle$  to  $|\varphi_\mu\rangle$  with probability  $w_\mu$ .

**Theorem 13.** Every transformation  $\mathcal{T} : |\psi\rangle \rightarrow \{(w_\mu, |\varphi_\mu\rangle)\}$  that does not increase  $\mathcal{C}$  on average, i.e. for which

$$\sum_\mu w_\mu \mathcal{C}(|\varphi_\mu\rangle) \leq \mathcal{C}(|\psi\rangle), \quad (58)$$

can be achieved by some  $Z_2$ -invariant operation.

(This theorem has an analog in entanglement theory; see theorem 2 in [35].)

**Proof.** Without loss of generality, we take the states to be in the standard form

$$\begin{aligned} |\psi\rangle &= \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle \\ |\varphi_\mu\rangle &= \sqrt{q_0^{(\mu)}}|0\rangle + \sqrt{q_1^{(\mu)}}|1\rangle, \end{aligned}$$

where  $p_0 \geq p_1$  and  $q_0^{(\mu)} \geq q_1^{(\mu)}$ .

We now define the state  $|\bar{\varphi}\rangle$  as

$$|\bar{\varphi}\rangle \equiv \sqrt{t_0}|0\rangle + \sqrt{t_1}|1\rangle, \quad (59)$$

where

$$t_1 \equiv \sum_\mu w_\mu q_1^{(\mu)}, \quad (60)$$

so that  $t_0 \geq t_1$ . Noting that  $\mathcal{C}(|\varphi_\mu\rangle) = 2q_1^{(\mu)}$  and  $\mathcal{C}(|\bar{\varphi}\rangle) = 2t_1$ , we infer from Eq. (60) that

$$\mathcal{C}(|\bar{\varphi}\rangle) = \sum_\mu w_\mu \mathcal{C}(|\varphi_\mu\rangle).$$

It then follows from Eq. (58) that  $\mathcal{C}(|\bar{\varphi}\rangle) \leq \mathcal{C}(|\psi\rangle)$ , which implies, by Thm. 10, that the transformation  $|\psi\rangle \rightarrow |\bar{\varphi}\rangle$  is achievable deterministically by  $Z_2$ -invariant operations. Therefore, we need only to show that we can generate the ensemble  $\{(w_\mu, |\varphi_\mu\rangle)\}$  starting from  $|\bar{\varphi}\rangle$ .

We now define the following set of positive  $Z_2$ -invariant Kraus operators:

$$K_\mu = \sqrt{\frac{w_\mu q_0^{(\mu)}}{t_0}}|0\rangle\langle 0| + \sqrt{\frac{w_\mu q_1^{(\mu)}}{t_1}}|1\rangle\langle 1|.$$

One can easily see that

$$\sum_\mu K_\mu^\dagger K_\mu = I,$$

and

$$K_\mu |\bar{\varphi}\rangle = \sqrt{w_\mu} |\varphi_\mu\rangle.$$

Hence, the combination of this measurement with the deterministic protocol  $|\psi\rangle \rightarrow |\bar{\varphi}\rangle$  realizes the required transformation  $\mathcal{T}$ . QED.

We are now in a position to prove Thm. 12.

**Proof of theorem 12.** Let  $F$  be an arbitrary frame-ness monotone. States in the standard form  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$  where  $p_0 \geq p_1$  are completely specified by  $p_1$ , or equivalently  $\mathcal{C}(|\psi\rangle) = 2p_1$ , and so  $F(|\psi\rangle)$  can be written as a function of  $\mathcal{C}(|\psi\rangle)$ , namely,  $F(|\psi\rangle) = f(\mathcal{C}(|\psi\rangle))$ . It remains only to show that  $f$  is a nondecreasing concave function.

If  $f$  was a decreasing function, then  $F$  would decrease under some transformation  $|\psi\rangle \rightarrow |\phi\rangle$  for which  $\mathcal{C}(|\psi\rangle) \geq \mathcal{C}(|\phi\rangle)$ . But by Thm. 10, every such transformation can be achieved deterministically and consequently  $F$  cannot decrease under this transformation if it is a monotone.

To prove that  $f$  is a concave function, it suffices to show that for all sets  $\{x_\mu\}$  such that  $x_\mu \in [0, 1]$  and all probability distributions  $w_\mu$ ,

$$\sum_\mu w_\mu f(x_\mu) \leq f\left(\sum_\mu w_\mu x_\mu\right). \quad (61)$$

Define a set of states  $\{|\varphi_\mu\rangle\}$  such that  $\mathcal{C}(|\varphi_\mu\rangle) = x_\mu$ , and another state  $|\psi\rangle$  such that  $\mathcal{C}(|\psi\rangle) = \sum_\mu w_\mu x_\mu$ , so that

$$\sum_\mu w_\mu \mathcal{C}(|\varphi_\mu\rangle) = \mathcal{C}(|\psi\rangle). \quad (62)$$

From Thm. 13 it follows that the transformation  $|\psi\rangle \rightarrow \{(w_\mu, |\varphi_\mu\rangle)\}$  can be achieved by  $Z_2$ -invariant operations. Given the presumed monotonicity of  $F$  under this transformation, we have

$$\sum_\mu w_\mu f(\mathcal{C}(|\varphi_\mu\rangle)) \leq f(\mathcal{C}(|\psi\rangle)). \quad (63)$$

Substituting Eq. (62) into Eq. (63), we obtain Eq. (61). QED.

### E. Stochastic single-copy transformations

We would like now to find the maximum possible probability to inter-convert one resource into another using only  $Z_2$ -invariant operations.

**Theorem 14.** *If the condition of Thm. 10 fails to be satisfied, so that  $\mathcal{C}(|\phi\rangle) > \mathcal{C}(|\psi\rangle)$ , then the maximum probability of transforming  $|\psi\rangle$  into  $|\phi\rangle$  using  $Z_2$ -invariant operations is*

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \frac{\mathcal{C}(|\psi\rangle)}{\mathcal{C}(|\phi\rangle)}.$$

If we express the states in the form where  $p_0 \geq p_1$  and  $q_0 \geq q_1$ , then the result may be expressed simply as  $P(|\psi\rangle \rightarrow |\phi\rangle) = p_1/q_1$ .

**Proof.** We begin by showing that  $P(|\psi\rangle \rightarrow |\phi\rangle) \leq \mathcal{C}(|\psi\rangle)/\mathcal{C}(|\phi\rangle)$ . The proof is by contradiction. If the transformation could be achieved with a probability  $P' > \mathcal{C}(|\psi\rangle)/\mathcal{C}(|\phi\rangle)$ , then the average value of  $\mathcal{C}$  after the measurement would be at least  $P'\mathcal{C}(|\phi\rangle) > \mathcal{C}(|\psi\rangle)$ , contradicting the fact that  $\mathcal{C}$  is an ensemble  $Z_2$ -frameness monotone and therefore nonincreasing under deterministic  $Z_2$ -invariant operations.

All that remains is to show that there is a protocol that saturates the inequality. The protocol is defined by a measurement with two possible outcomes corresponding to Kraus operators

$$K_0 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K_0^\perp = \begin{pmatrix} \sqrt{1-x^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\text{where } x = \sqrt{\frac{q_0 p_1}{q_1 p_0}}.$$

One verifies that this is a possible measurement operation by noting that  $K_0^\dagger K_0$ ,  $K_0^{\perp\dagger} K_0^\perp \geq 0$  and  $K_0^\dagger K_0 + K_0^{\perp\dagger} K_0^\perp = I$ . The operation is  $Z_2$ -invariant, because both  $K_0$  and  $K_0^\perp$  are of the form specified in lemma 9. Finally, by noting that

$$K_0 = \sqrt{\frac{p_1}{q_1}} \begin{pmatrix} \sqrt{q_0/p_0} & 0 \\ 0 & \sqrt{q_1/p_1} \end{pmatrix}$$

it is straightforward to see that if the  $K_0$  outcome occurs, the state collapses to  $|\phi\rangle$  with probability  $w_0 = |K_0|\psi\rangle| = p_1/q_1 = \mathcal{C}(|\psi\rangle)/\mathcal{C}(|\phi\rangle)$ . QED.

### F. Stochastic $Z_2$ -frameness monotones

By Thm. 14, the only instance of the probability of a transformation  $|\psi\rangle \rightarrow |\varphi\rangle$  being zero is if  $\mathcal{C}(|\psi\rangle) = 0$  while  $\mathcal{C}(|\varphi\rangle) \neq 0$ , but this is just the obviously impossible case of a transformation from a state  $|\psi\rangle = |0\rangle$  or  $|1\rangle$  that is  $Z_2$ -invariant to a state  $|\varphi\rangle$  that is  $Z_2$ -noninvariant. It follows that the only *stochastic*  $Z_2$ -frameness monotone is the trivial one – the number of parity eigenstates receiving nonzero weight. We call this quantity the *chiral spectrum cardinality*. It is clearly analogous to the Schmidt number, which is a stochastic entanglement monotone.

Note, however, that whereas two entangled states may differ in Schmidt number, so that the conversion of one to the other can only be achieved in one direction, all pairs of states with nonzero  $Z_2$ -frameness have chiral spectrum cardinality of two, and therefore can be stochastically converted one to the other in either direction. In this sense, the restriction of the  $Z_2$ -SSR allows more possibilities for resource interconversion than the restriction of LOCC for pure bipartite states.

### G. Asymptotic transformations

Consider a state that has a decomposition into even and odd parity states of the form  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$ .

**Theorem 15.** *The unique (modulo normalization) measure of  $Z_2$ -frameness for pure states is*

$$F^\infty(|\psi\rangle) = -\log |p_0 - p_1|. \quad (64)$$

**Proof.** We assume  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$  with  $p_0 \geq p_1$ . Note first that

$$|\psi\rangle^{\otimes N} = \sum_{m=0}^N \sqrt{p_0}^m \sqrt{p_1}^{N-m} \sum |0\rangle^{\otimes m} |1\rangle^{\otimes N-m}$$

where the final sum is over all the ways of having  $m$  systems in state  $|0\rangle$  and  $N-m$  in state  $|1\rangle$ . It is useful to decompose this into unnormalized states of even and odd parity,

$$|\psi\rangle^{\otimes N} = |\tilde{\chi}_{\text{even}}\rangle + |\tilde{\chi}_{\text{odd}}\rangle,$$

where  $|\tilde{\chi}_{\text{even}}\rangle$  contains the terms where  $N-m$  is even, and  $|\tilde{\chi}_{\text{odd}}\rangle$  contains the terms where  $N-m$  is odd. Noting that

$$\begin{aligned} ||\tilde{\chi}_{\text{even}}\rangle| &= \sum_{m|N-m \text{ even}} \binom{N}{m} p_0^m p_1^{N-m} \\ &= \frac{1}{2} ((p_0 + p_1)^N + (p_0 - p_1)^N) \\ &= \frac{1}{2} (1 + (p_0 - p_1)^N), \end{aligned}$$



and bearing in mind that the normalized states  $|\tilde{\chi}_{\text{even}}\rangle/|\tilde{\chi}_{\text{even}}|$  and  $|\tilde{\chi}_{\text{odd}}\rangle/|\tilde{\chi}_{\text{odd}}|$  can be transformed by a  $Z_2$ -invariant unitary into  $|0\rangle$  and  $|1\rangle$ , we infer that the standard form of  $|\psi\rangle^{\otimes N}$  is

$$|\psi\rangle^{\otimes N} = \sqrt{r_0}|0\rangle + \sqrt{r_1}|1\rangle. \quad (65)$$

where

$$r_0 \equiv \frac{1}{2} + \frac{1}{2}(p_0 - p_1)^N, \quad (66)$$

$$r_1 \equiv \frac{1}{2} - \frac{1}{2}(p_0 - p_1)^N. \quad (67)$$

Note that because we have assumed  $p_0 \geq p_1$ , it follows that  $r_0 \geq r_1$ .

Suppose that the target state has the standard form  $|\phi\rangle = \sqrt{q_0}|0\rangle + \sqrt{q_1}|1\rangle$  where  $q_0 \geq q_1$ . We can find an analogous expression to Eq. (65) for  $|\phi\rangle^{\otimes M}$ . The condition for the existence of a reversible transformation between  $|\psi\rangle^{\otimes N}$  and  $|\phi\rangle^{\otimes M}$  is simply the condition that their standard forms be equal, namely,

$$(p_0 - p_1)^N = (q_0 - q_1)^M. \quad (68)$$

Suppose that neither  $|\psi\rangle$  nor  $|\phi\rangle$  is the degenerate state  $|+\rangle$ , that is, assume that  $p_0 \neq p_1$  and  $q_0 \neq q_1$ . Taking the absolute value and logarithm on both sides of the condition, we obtain

$$\lim_{N \rightarrow \infty} \frac{M}{N} = \frac{\log(p_0 - p_1)}{\log(q_0 - q_1)}.$$

It follows from Eq. (25) that the measure that determines the rate of asymptotic reversible interconversion is

$$F^\infty(|\psi\rangle) = \mathcal{N} \log(p_0 - p_1) \quad (69)$$

for some normalization factor  $\mathcal{N}$ .

For a pure state  $|\psi\rangle$  that is not in the standard form, so that  $p_0 < p_1$ , the same reasoning implies that  $F^\infty(|\psi\rangle) = \mathcal{N} \log(p_1 - p_0)$ . It follows that  $F^\infty(|\psi\rangle) = \mathcal{N} \log|p_0 - p_1|$  provides a measure for an arbitrary pure state. The normalization  $\mathcal{N}$  is a conventional choice which we take to be  $-1$ . The base of the logarithm is also a conventional choice which we take to be 2. These choices are discussed below.

Finally, we need to consider the degenerate cases. If  $q_0 \neq q_1$  but  $p_0 = p_1$ , then the condition for  $|\psi\rangle^{\otimes N} \leftrightarrow |\phi\rangle^{\otimes M}$ , Eq. 68, becomes

$$(q_0 - q_1)^M = 0.$$

Consequently, for any finite value of  $N$ , the value of  $M$  must become arbitrarily large to satisfy the condition, in other words, the rate becomes infinite,  $M/N \rightarrow \infty$ . This means that the degenerate state  $|+\rangle$  can be transformed reversibly into an arbitrarily large number of copies of any nondegenerate state.

If  $p_0 \neq p_1$  but  $q_0 = q_1$ , then Eq. 68 becomes

$$(p_0 - p_1)^N = 0.$$

Consequently, for any value of  $M$ , we have  $M/N = 0$ . This means that the degenerate state  $|+\rangle$  cannot be obtained reversibly from any number of copies of a nondegenerate state.

Both of these facts are captured by defining  $F^\infty(|+\rangle) = \infty$  for the degenerate state  $|+\rangle \equiv 2^{-1/2}(|0\rangle + |1\rangle)$ . The expression for  $F^\infty$  in Eq. (64) can therefore be taken to apply even to the degenerate state. QED.

It is worth noting that  $F^\infty(|+\rangle) = \infty$  implies that  $N$  copies of  $|+\rangle$  can be transformed to *one* copy of  $|+\rangle$  and vice-versa. Therefore, the state  $|+\rangle$  alleviates completely the  $Z_2$ -SSR. Furthermore, as described in the proof, the asymptotic rate with which one can produce  $|+\rangle$  given any state that is not  $|+\rangle$  is zero. The state  $|+\rangle$  is therefore a very special sort of resource – it is sufficient to completely lift the SSR and no amount of any lesser resource can substitute for it. Although it is the only distinguished state in the  $Z_2$  resource theory that has nonzero frameness, the  $|+\rangle$  state cannot play a role analogous to the one played by the singlet state in entanglement distillation, because we cannot distill any amount of  $|+\rangle$  from any other state.

If not  $|+\rangle$ , then what *does* make a good choice of standard resource against which to judge the strength of any given state? Any nondegenerate state with nonzero frameness will do. We adopt a convention that makes the expression for  $F^\infty$  particularly simple. First, we assume the logarithm to be base 2. Second, we take the normalization factor  $\mathcal{N}$  introduced in the proof to be  $\mathcal{N} = -1$ , so that  $F^\infty$  has the form presented in Eq. (64). The negative sign is chosen to ensure that the measure is positive. The unit magnitude of the normalization implies that the state having unit frameness,  $F^\infty = 1$ , is the one for which  $p_0 = 3/4$ , that is,  $\sqrt{3}/2|0\rangle + 1/\sqrt{2}|1\rangle$ . The measure  $F^\infty(|\psi\rangle)$  then quantifies the number of states of this form that one can distill from  $|\psi\rangle$  asymptotically.

We may, of course, express the measure of  $Z_2$ -frameness entirely in terms of  $p_0$ ,  $F^\infty(|\psi\rangle) \equiv -\log|2p_0 - 1|$ . A plot of  $F^\infty$  as a function of  $p_0$  is provided in Fig. 4.

Finally, we note that although  $F^\infty$  is a deterministic monotone (by virtue of quantifying the asymptotic rate of conversion and the fact that any such measure is a deterministic monotone, as shown in Sec. IIF), it is *not* an *ensemble* monotone.  $F^\infty$  is related to  $\mathcal{C}$  by

$$F^\infty(|\psi\rangle) = -\log(1 - \mathcal{C}(|\psi\rangle)),$$

where we drop the absolute value because  $\mathcal{C}(|\psi\rangle) \leq 1$ . Given that the second derivative with respect to  $x$  of  $-\log(1 - x)$  is always positive,  $F^\infty$  is a convex function of  $\mathcal{C}$  and consequently is not an ensemble monotone.

There is another, more intuitive, way to see that  $F^\infty$  cannot be an ensemble monotone. Theorem 14 implies that a state  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$  with nonzero

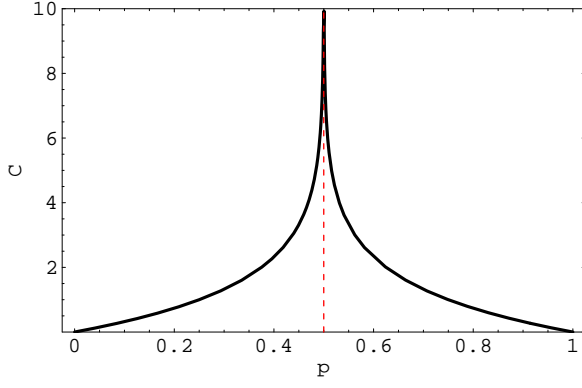


FIG. 4: A plot of the asymptotic measure  $F^\infty$  of  $Z_2$ -frameness as a function of the probability  $p_0$  of the state having even parity.

frameness can always be converted to the state  $|+\rangle$  with nonzero probability (indeed, the probability is simply  $\mathcal{C}(|\psi\rangle)$ ). It follows that  $|\psi\rangle$  can be converted to an ensemble of states that assign nonzero weight to  $|+\rangle$ . However, given that  $F^\infty(|+\rangle)$  is unbounded, the average fidelity for this ensemble will also be unbounded. It follows that the average value of  $F^\infty$  can be increased using  $Z_2$ -invariant operations and so it fails to be an ensemble monotone.

We see that the property of ensemble monotonicity need not hold for the unique asymptotic measure of a resource. This result calls into question the widespread tendency to require any measure of a resource (such as entanglement or frameness) to be an ensemble monotone. As discussed in Sec. II C, to be operationally well-motivated, it may suffice for a measure of frameness to be a *deterministic* monotone rather than an *ensemble* monotone, which is precisely what occurs in the case of  $F^\infty$ .

We end this section by showing that the measure  $F^\infty(\psi)$  has a nice additivity property.

**Proposition 16.**  $F^\infty(\psi)$  is strongly additive.

**Proof:** Let  $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{p_1}|1\rangle$  with  $p_0 \geq p_1$  and  $|\phi\rangle = \sqrt{q_0}|0\rangle + \sqrt{q_1}|1\rangle$  with  $q_0 \geq q_1$ . The tensor product of these two states is:

$$|\psi\rangle|\phi\rangle = \sqrt{r_0}|\chi_0\rangle + \sqrt{r_1}|\chi_1\rangle,$$

where  $r_0 \equiv p_0q_0 + p_1q_1$ ,  $r_1 \equiv p_0q_1 + p_1q_0$ ,  $|\chi_0\rangle \equiv (\sqrt{p_0q_0}|00\rangle + \sqrt{p_1q_1}|11\rangle)/\sqrt{r_0}$  and  $|\chi_1\rangle \equiv (\sqrt{p_0q_1}|01\rangle + \sqrt{p_1q_0}|10\rangle)/\sqrt{r_1}$ . Noting that  $r_0 \geq r_1$  (because  $r_0 - r_1 = (p_0 - p_1)(q_0 - q_1) \geq 0$ ), and noting that we can transform  $|\chi_0\rangle \rightarrow |0\rangle$  and  $|\chi_1\rangle \rightarrow |1\rangle$  by a  $Z_2$ -invariant unitary, we see that  $\sqrt{r_0}|0\rangle + \sqrt{r_1}|1\rangle$  is the standard form of the tensor product state. Therefore,

$$\begin{aligned} F^\infty(|\psi\rangle|\phi\rangle) &= -\log|r_0 - r_1| \\ &= -\log|(p_0 - p_1)(q_0 - q_1)| \\ &= -\log|p_0 - p_1| - \log|q_0 - q_1| \\ &= F^\infty(|\psi\rangle) + F^\infty(|\phi\rangle). \end{aligned}$$

QED.

## V. RESOURCE THEORY OF THE SU(2)-SSR

### A. Frames for orientation

A reference frame for orientation, commonly called a Cartesian frame, is associated with  $SO(3)$ , the rotation group. An element of  $SO(3)$  can be given, for instance, by specifying three Euler angles. We will represent it instead by a vector  $\vec{\theta}$ , representing a rotation by  $\theta$  about the axis  $\hat{\theta} = \vec{\theta}/\theta$ . We will extend the group of rotations  $SO(3)$  to the group  $SU(2)$  to allow for spinor representations. The representation  $T$  of  $SU(2)$  on a Hilbert space  $\mathcal{H}$  determines how the quantum system transforms under rotations,

$$T(\vec{\theta}) = \exp(i\vec{\theta} \cdot \hat{\mathbf{J}}),$$

where  $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$  is the angular momentum operator.

The states that are  $SU(2)$ -invariant, which we may also describe as rotationally invariant, are those satisfying

$$T(\vec{\theta})\rho T^\dagger(\vec{\theta}) = \rho \quad \forall \vec{\theta} \in SU(2).$$

This is equivalent to the condition

$$[\rho, \hat{J}_x] = [\rho, \hat{J}_y] = [\rho, \hat{J}_z] = 0.$$

It is useful to decompose the Hilbert space as

$$\mathcal{H} = \bigoplus_j \mathcal{M}_j \otimes \mathcal{N}_j, \quad (70)$$

where  $j \in \{0, 1/2, 1, 3/2, \dots\}$  is the angular momentum quantum number, the  $\mathcal{M}_j$  carry irreducible representations of  $SU(2)$ , and the  $\mathcal{N}_j$  are the multiplicity spaces, carrying the trivial representation of  $SU(2)$ .

Relative to this decomposition, the  $SU(2)$ -invariant states have the form [6]

$$\rho = \sum_j p_j \frac{I_{\mathcal{M}_j}}{\dim(\mathcal{M}_j)} \otimes \rho_{\mathcal{N}_j},$$

where  $I_{\mathcal{M}_j}$  is the identity operator on  $\mathcal{M}_j$ ,  $\dim(I_{\mathcal{M}_j})$  is the dimension of  $\mathcal{M}_j$ ,  $p_j$  is a probability distribution over  $j$ , and  $\rho_{\mathcal{N}_j}$  is an arbitrary density operator on  $\mathcal{N}_j$ .

We again focus our attention on pure states. Relative to the decomposition of Eq. (70), a general pure state can be written as

$$|\psi\rangle = \sum_{j,m,\beta} c_{jm\beta} |j,m\rangle \otimes |j,\beta\rangle,$$

where the  $|j,m\rangle$  form a basis of  $\mathcal{M}_j$  and the  $|j,\beta\rangle$  form a basis of  $\mathcal{N}_j$ . It transforms under  $SU(2)$  as

$$T(\vec{\theta})|\psi\rangle = \sum_{j,m,\beta} \left( T_j(\vec{\theta}) |j,m\rangle \right) \otimes |j,\beta\rangle,$$

where  $T_j(\vec{\theta})$  is the  $j$ th irreducible unitary representation of  $SU(2)$ .

Again, any operation on the  $\mathcal{N}_j$  can be achieved under the  $SU(2)$ -SSR, therefore an arbitrary pure state can always be transformed into the standard form

$$|\psi\rangle = \sum_{j,m} c_{jm} |j, m\rangle,$$

and we presume this form henceforth. Effectively, we are confining ourselves to the Hilbert space  $\mathcal{H}' = \bigoplus_j \mathcal{M}_j = \text{span}\{|j, m\rangle\}_{j,m} \subseteq \mathcal{H}$ .

We can now write  $SU(2)$ -invariant states simply as

$$\rho = \sum_j p_j \Pi_j \quad (71)$$

where  $\Pi_j = \sum_m |j, m\rangle \langle j, m|$ .

It is interesting to note that the only pure state that is  $SU(2)$ -invariant is  $|j=0, m=0\rangle$ , because it is the only one for which the density operator has the form of Eq. (71). Consequently, it is the only pure state that is free under the  $SU(2)$ -SSR – every other pure state is a resource. To see how these resources can be manipulated, we must derive the form of  $SU(2)$ -invariant operations.

## B. $SU(2)$ -invariant operations

We now apply Lemma 1 to the problem of characterizing the  $SU(2)$ -invariant operations. Recall that the irreducible representations of  $SU(2)$  are labeled by the set of nonnegative integers and half-integers  $J \in \{0, 1/2, 1, 3/2, \dots\}$  and are each of dimension  $2J+1$ . The  $J$ th irreducible unitary representation of  $SU(2)$ ,  $u^{(J)} : SU(2) \rightarrow \mathbb{C}^{2J+1}$ , has matrix elements

$$u_{MM'}^{(J)}(\vec{\theta}) \equiv \langle J, M | e^{-i\vec{\theta} \cdot \hat{\mathbf{J}}} | J, M' \rangle, \quad (72)$$

where  $|J, M\rangle$  is the joint eigenstate of  $\hat{J}^2$  and  $J_z$  with eigenvalues  $J(J+1)$  and  $\hbar M$ .

It then follows from Lemma 1 that an  $SU(2)$ -invariant operation has Kraus operators  $K_{J,M,\alpha}$ , labeled by an irrep  $J$ , a basis element  $M$ , and a multiplicity index  $\alpha$ , satisfying

$$e^{i\vec{\theta} \cdot \hat{\mathbf{J}}} K_{J,M,\alpha} e^{-i\vec{\theta} \cdot \hat{\mathbf{J}}} = \sum_{M'} u_{MM'}^{(J)}(\vec{\theta}) K_{J,M',\alpha}, \quad \forall \vec{\theta} \in SU(2). \quad (73)$$

The set of operators  $\{K_{J,M,\alpha} | M\}$  for fixed  $J$  and  $\alpha$  is sometimes called a *spherical tensor* of rank  $J$  (see e.g. p. 569 in [37]).

The simplest case to consider is the  $J=0$  irrep, which is the trivial representation

$$u^{(0)}(\vec{\theta}) = 1.$$

A Kraus operator associated with this irrep, denoted  $K_{0,0,\alpha}$ , satisfies

$$e^{i\vec{\theta} \cdot \hat{\mathbf{J}}} K_{0,0,\alpha} e^{-i\vec{\theta} \cdot \hat{\mathbf{J}}} = K_{0,0,\alpha}, \quad \forall \vec{\theta} \in SU(2).$$

It follows that  $K_{0,0,\alpha}$  is an  $SU(2)$ -invariant operator and therefore has the form

$$K_{0,0,\alpha} = \sum_j c_j^{(\alpha)} \Pi_j, \quad (74)$$

where  $\Pi_j = \sum_{m=-j}^j |j, m\rangle \langle j, m|$  and the  $c_j^{(\alpha)}$  are complex numbers.

By taking derivatives of Eq. (73) relative to the different components of  $\vec{\theta}$  and then setting  $\vec{\theta} = 0$ , one finds that a Kraus decomposition  $\{K_{J,M,\alpha}\}$  can always be found that satisfies

$$[\hat{J}_z, K_{J,M,\alpha}] = \hbar M K_{J,M,\alpha} \quad (75)$$

$$[\hat{J}_{\pm}, K_{J,M,\alpha}] = \hbar \sqrt{J(J+1) - M(M \pm 1)} K_{J,M \pm 1, \alpha} \quad (76)$$

where  $\hat{J}_{\pm} \equiv \frac{1}{\sqrt{2}}(\hat{J}_x \pm \hat{J}_y)$  are the angular momentum raising and lowering operators.

The Wigner-Eckart theorem (see e.g. p. 239 in [17]) famously specifies the form of the spherical tensor operators of rank  $J$ : a set of operators  $\{K_{J,M,\alpha}\}$  satisfy Eqs. (75) and (76) if and only if

$$\langle j', m' | K_{J,M,\alpha} | j, m \rangle \quad (77)$$

$$= (-1)^{j'-m'} \begin{pmatrix} j' & J & j \\ -m' & M & m \end{pmatrix} \langle j' || K_{J,\alpha} || j \rangle, \quad (78)$$

where  $f_{J,\alpha}(j', j) \equiv \langle j' || K_{J,\alpha} || j \rangle$  does not depend on  $m, m'$  or  $M$ . Note that for *any* choice of  $f_{J,\alpha}(j', j)$ , the matrices  $K_{J,M,\alpha}$  as defined above satisfy the commutation relations in Eqs. (75) and (76).

We therefore conclude that an  $SU(2)$ -invariant operation admits a Kraus decomposition  $\{K_{J,M,\alpha}\}$  where

$$\langle j', m' | K_{J,M,\alpha} | j, m \rangle \quad (79)$$

$$= (-1)^{j'-m'} \begin{pmatrix} j' & J & j \\ -m' & M & m \end{pmatrix} f_{J,\alpha}(j', j), \quad (80)$$

for some choice of  $f_{J,\alpha}(j', j)$ . We require that  $\sum_{J,\alpha} |f_{J,\alpha}(j', j)|^2 \leq 2j+1$  for all  $j, j'$ .

Recalling that Wigner's  $3j$  symbols are defined in terms of Clebsch-Gordan coefficients by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1, m_1, j_2, m_2 | j_3, -m_3 \rangle, \quad (81)$$

and that

$$(j_1, m_1, j_2, m_2 | j, m) = \delta_{m, m_1+m_2} (j_1, m_1, j_2, m_2 | j, m_1+m_2),$$

we conclude that  $\langle j', m' | K_{J,M,\alpha} | j, m \rangle$  is only nonzero if  $m = m' - M$ . We summarize the result by the following lemma.

**Lemma 17.** *An arbitrary  $SU(2)$ -invariant operation on  $\mathcal{B}(\mathcal{H}')$  admits a Kraus decomposition  $\{K_{J,M,\alpha}\}$ , where*

$J \in \{0, 1/2, 1, 3/2, \dots\}$ ,  $M \in \{-J, \dots, J\}$  and  $\alpha$  is an integer, such that

$$K_{J,M,\alpha} = \sum_{j'=0,1/2,1,\dots} \sum_{m=-j'}^{j'} \sum_{j=|J-j'|}^{J+j'} (-1)^{j'-m} \times \begin{pmatrix} j' & J & j \\ -m & M & m-M \end{pmatrix} \times f_{J,\alpha}(j', j) |j', m\rangle \langle j, m-M|. \quad (82)$$

where the  $2 \times 3$  matrix is a Wigner  $3j$  symbol and the function  $f_{J,\alpha}(j', j)$  does not depend on  $m$  or  $M$ , and satisfies  $\sum_{J,j',\alpha} |f_{J,\alpha}(j', j)|^2 \leq 2j+1$  for all  $j$ , with equality if the operation is trace-preserving.

We recover Eq. (74) by noting that for  $J = M = 0$ , we have

$$\begin{pmatrix} j' & 0 & j \\ -m & 0 & m \end{pmatrix} = \frac{(-1)^{j'-m}}{\sqrt{2j+1}} \delta_{j,j'},$$

and consequently

$$K_{0,0,\alpha} = \sum_{j=0,1/2,1,\dots} \frac{f_\alpha(j)}{\sqrt{2j+1}} \Pi_j,$$

for some amplitudes  $f_\alpha(j)$ .

A different characterization of SU(2)-invariant operations is provided in Boileau *et al.* [36], but we shall not make use of it here. Determining the connection between it and the characterization provided above is a subject for future research.

### 1. SU(2)-invariant unitaries

In addition to the unitaries defined on the multiplicity spaces  $\mathcal{N}_j$ , there are SU(2)-invariant unitaries on the subspace  $\mathcal{H}' = \text{span}\{|j, m\rangle\}_{j,m} = \bigoplus_j \mathcal{M}_j$ . Unitary operations have only a single Kraus operator, so there are unitaries among the  $J = 0$  irreducible SU(2)-invariant operations. The single (unitary) Kraus operator for such an operation has the form  $K_{0,0} = \sum_j c_j \Pi_j$ , where  $|c_j^{(\alpha)}| = 1$  for all  $j$ . These operations on  $\mathcal{H}'$  simply change the relative phases between the  $\mathcal{M}_j$  subspaces.

## C. A restricted set of SU(2) frame states

From the very outset, it is clear that there will be several distinct sorts of resources under the SU(2)-SSR. To see this, note that there is a distinction between a quantum Cartesian frame (a state that picks out a triad of orthogonal spatial directions) and a symmetric quantum direction indicator (a state that only picks out a single direction in space and is symmetric under rotations about that direction). Symmetric quantum direction indicators for distinct directions are clearly inequivalent resources

because to transform one to the other would require the ability to rotate about some third axis, and the latter operation is forbidden under the SU(2)-SSR. It then follows that a symmetric quantum direction indicator is not equivalent to a quantum Cartesian frame because the latter can only be built out of a *pair* of the former for distinct directions. So we can already see that two resources under the SU(2)-SSR need not be interconvertible.

The general problem of the transformation of pure resource states under the SU(2)-SSR appears to be very difficult and we do not attempt to solve it completely here. Rather, we restrict our attention to a subset of states. To define this set, recall the definition of an SU(2)-coherent state. It is a highest weight state  $|j, m=j\rangle_{\hat{n}}$  where  $\hat{n}$  denotes the quantization axis. Now define  $\mathcal{H}_{\hat{n}} \equiv \text{span}\{|j, j\rangle_{\hat{n}} | j=0, 1/2, 1, \dots\}$ , the subspace of  $\mathcal{H}'$  consisting of all linear combinations of SU(2)-coherent states associated with the same quantization axis. Note that every state in  $\mathcal{H}_{\hat{n}}$  except one is a resource under the SU(2)-SSR. The exception is the singlet  $|0, 0\rangle$ . Consequently, we can define a set of resource states by removing the singlet,  $\mathcal{C}_{\hat{n}} \equiv \mathcal{H}_{\hat{n}} - \text{span}\{|0, 0\rangle\}$ . Note that only the states of the form  $|j, j\rangle_{\hat{n}}$  are symmetric direction indicators for  $\hat{n}$  because any linear combination of these fails to be invariant under rotations about  $\hat{n}$  (the physical interpretation of states in  $\mathcal{C}_{\hat{n}}$  is unclear at present). We are finally in a position to define the full set of quantum reference states with which we will concern ourselves here. It is

$$\mathcal{C} \equiv \bigcup_{\hat{n} \in S_2} \mathcal{C}_{\hat{n}},$$

the union of the  $\mathcal{C}_{\hat{n}}$  for all directions  $\hat{n}$  on the unit sphere  $S_2$ , that is, all choices of quantization axis. Note that the set  $\mathcal{C}$  is not exhaustive. States in  $\mathcal{H}'$  assigning nonzero amplitude to any  $|j, m\rangle$  with  $|m| < j$  are excluded.

We will show that an element of  $\mathcal{C}_{\hat{n}}$  and an element of  $\mathcal{C}_{\hat{m}}$  where  $\hat{n} \neq \hat{m}$  are inequivalent resources in the sense that one cannot convert one to the other, not even with probability less than unity.

It follows that there is a continuous infinity of different types of resources under the SU(2)-SSR. This is similar to what occurs for pure state entanglement for four qubits. In this sense, the resource theory for SU(2) frames is degenerate. Nonetheless, one can still ask what transformations are possible *within* a class  $\mathcal{C}_{\hat{n}}$  and indeed, a nontrivial structure is found to which we turn in subsequent sections. Here, we prove the existence of the distinct classes.

### 1. Proof of the existence of inequivalent classes within the restricted set

An arbitrary state in  $\mathcal{C}_{\hat{n}}$  can be written as  $|\psi\rangle = \sum_j c_j |j, j\rangle$ . However, it can always be transformed, by

SU(2)-invariant unitaries, into the standard form

$$|\psi\rangle = \sum_j \sqrt{p_j} |j, j\rangle,$$

where  $0 \leq p_j \leq 1$  and  $\sum_j p_j = 1$ . Note further that  $|j, j\rangle \otimes |j', j'\rangle = |j + j', j + j'\rangle$  (as a straightforward calculation of Clebsch-Gordan coefficients confirms) so that multiple systems with states drawn from  $\mathcal{C}_{\hat{n}}$  can also be represented in the standard form. We assume this form in the following.

We wish to determine which, if any, SU(2)-invariant operations  $\mathcal{E}$  can take a state  $|\psi\rangle \in \mathcal{H}_{\hat{n}}$  to a pure state,  $\mathcal{E}(|\psi\rangle\langle\psi|) = \lambda |\phi\rangle\langle\phi|$ . We do not assume that  $|\phi\rangle \in \mathcal{H}_{\hat{n}}$  (although it will be shown that this is the case for SU(2)-invariant operations that take pure states to pure states).

Every SU(2)-invariant operation can be written as a convex sum of irreducible SU(2)-invariant operations,  $\mathcal{E} = \sum_{J,\alpha} w_{J,\alpha} \mathcal{E}_{J,\alpha}$ , where the irreducible operations are labelled by the irreps  $J$  and a multiplicity index  $\alpha$ . It suffices therefore to identify, for a given  $J$ , which irreducible SU(2)-invariant operations  $\mathcal{E}_J$  can take a pure state  $|\psi\rangle \in \mathcal{H}_{\hat{n}}$  to another pure state. Recalling Eq. (74), an irreducible SU(2)-invariant operation  $\mathcal{E}_0$  associated with  $J = 0$  has a single Kraus operator  $K_{0,0} = \sum_j c_j \Pi_j$  which clearly takes  $|\psi\rangle$  to a pure state within  $\mathcal{H}_{\hat{n}}$ .

The interesting case is  $J > 0$ . A given  $\mathcal{E}_J$  has Kraus operators  $\{K_{J,M} | M \in \{-J, \dots, J\}\}$  where the  $K_{J,M}$  satisfy Eq. (82). The only freedom we have is in the variation of the function  $f_J(j, j')$ . Our question, therefore, is: for what choice of  $f_J(j, j')$  can one achieve

$$\sum_M K_{J,M} |\psi\rangle\langle\psi| K_{J,M}^\dagger = \lambda |\phi\rangle\langle\phi|,$$

or equivalently,

$$K_{J,M} |\psi\rangle = h(J, M) |\phi\rangle \text{ for all } M \in \{-J, \dots, J\}, \quad (83)$$

where the functions  $h(J, M)$  satisfy

$$\sum_{M=-J}^J |h(J, M)|^2 = \lambda.$$

**Theorem 18.** *If the irreducible SU(2)-invariant map  $\mathcal{E}_J$  takes a pure state  $|\psi\rangle \in \mathcal{H}_{\hat{n}}$  to another pure state,  $|\phi\rangle$ , then the restriction of  $\mathcal{E}_J$  to  $\mathcal{H}_{\hat{n}}$  must have Kraus operators of the form*

$$\begin{aligned} K_{J,M} &= 0 \text{ for } M \in \{-J+1, \dots, J\} \\ K_{J,-J} &= \sum_{j \geq J} c_j^{(J)} |j-J, j-J\rangle\langle j, j|, \end{aligned} \quad (84)$$

where the  $c_j^{(J)}$  are complex coefficients satisfying  $|c_j^{(J)}|^2 \leq 1$  for all  $j$ , with equality if the operation is trace-preserving.

Note that due to the form of  $\mathcal{E}_J$ , the output state  $|\phi\rangle$  is always in  $\mathcal{H}_{\hat{n}}$ . Consequently, SU(2)-invariant maps cannot transform a pure state inside  $\mathcal{H}_{\hat{n}}$  to one outside  $\mathcal{H}_{\hat{n}}$  with any probability.

**Proof.** Suppose that  $|\psi\rangle = \sum_j \sqrt{p_j} |j, j\rangle$ . From Eq. (82), we infer that

$$K_{J,M} |\psi\rangle = \sum_{j'} \sum_{j=|J-j'|}^{J+j'} a_{j,j'}^{(J,M)} |j', j+M\rangle \quad (85)$$

where the amplitudes are

$$\begin{aligned} a_{j,j'}^{(J,M)} &\equiv (-1)^{j'-(j+M)} \begin{pmatrix} j' & J & j \\ -(j+M) & M & j \end{pmatrix} \\ &\times f_J(j', j) \sqrt{p_j}. \end{aligned} \quad (86)$$

Recalling the definition of the Wigner  $3j$  symbol in terms of Clebsch-Gordan coefficients (Eq. (81)), we note that necessary conditions for a Wigner  $3j$  symbol to be nonzero,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \neq 0,$$

are

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2, \quad (87)$$

$$|m_1| \leq j_1, \quad (88)$$

$$|m_2| \leq j_2. \quad (89)$$

For the particular  $3j$  symbol appearing in Eq. (86) to be nonzero, the necessary conditions are

$$|J - j'| \leq j \leq j' + J \quad (90)$$

$$|-(j+M)| \leq j'$$

$$|M| \leq J.$$

We point out that these conditions are also *sufficient* for the  $3j$  symbol that appears in Eq. (86) to be non-zero (see the formula C. 24 in p. 1059 of ref. [37]). The latter two constraints can be written as bounds on  $M$ ,

$$-(j' + j) \leq M \leq j' - j, \quad (91)$$

$$-J \leq M \leq J. \quad (92)$$

From the first inequality in Eq. (90), we can deduce that  $-(j+j') \leq -J$  which implies that the larger of the two lower bounds on  $M$  is the second. Similarly, we can deduce from this inequality that  $j' - j \leq J$  which implies that the smaller of the two upper bounds on  $M$  is the first. All told, we have

$$-J \leq M \leq j' - j. \quad (93)$$

Consider first the case wherein the  $3j$  symbol is nonzero for only a *single* value of  $M$ . To ensure that

this is the case, we must ensure that the lower and upper bounds on  $M$  coincide, that is, we must take

$$M = j' - j = -J.$$

This constraint can be enforced by choosing

$$f_J(j', j) = \delta_{j', j-J} f_J(j).$$

By substituting this choice of  $f_J(j', j)$  into Eq. (85), we obtain Eq. (84), the allowed form of  $\mathcal{E}_J$  given in the theorem. To prove the theorem, we must show that this is the *only* possible form that  $f_J(j', j)$  can take.

We do so by assuming the contrary and deriving a contradiction. Suppose that  $f_J(j', j) \neq 0$  for some triple of values  $J, j', j$  satisfying  $j' - j > -J$  (so that the upper and lower bounds in Eq. (93) do *not* coincide) and  $j \geq |J - j'|$  (so that the 3j-symbols are nonzero). Note that for fixed  $J$  and  $j'$ , the  $j$  value for such a triple is in the range

$$|J - j'| \leq j < J + j'.$$

For such distinguished triples,  $a_{jj'}^{(J,M)} \neq 0$  and consequently  $K_{J,M}|\psi\rangle \neq 0$ , for all  $M$  in the range  $-J \leq M \leq j' - j$ , which includes more than one value.

Suppose  $M_1, M_2$  are two distinct values in the given range. Equation (83) implies that  $K_{J,M_1}|\psi\rangle$  and  $K_{J,M_2}|\psi\rangle$  must be proportional to each other. It follows from Eq. (85) that for  $J, j', j$  a distinguished triple, the state  $K_{J,M_1}|\psi\rangle$  assigns nonzero amplitude to the term  $|j', m'_1\rangle$  with  $m'_1 \equiv j + M_1$ . Similarly,  $K_{J,M_2}|\psi\rangle$  assigns nonzero amplitude to the term  $|j', m'_2\rangle$  with  $m'_2 \equiv j + M_2$ . However, given the assumed proportionality of  $K_{J,M_1}|\psi\rangle$  and  $K_{J,M_2}|\psi\rangle$ , we must also have the former term in  $K_{J,M_2}|\psi\rangle$  and the latter term in  $K_{J,M_1}|\psi\rangle$ . It follows that there must be another value of  $j$ , call it  $j_{\max}$ , such that  $K_{J,M_1}|j_{\max}, j_{\max}\rangle$  assigns nonzero amplitude to  $|j', m'_2\rangle$  (hence  $f_J(j', j_{\max}) \neq 0$ ) and a third value of  $j$ , call it  $j_{\min}$ , such that  $K_{J,M_2}|j_{\min}, j_{\min}\rangle$  assigns nonzero amplitude to  $|j', m'_1\rangle$  (hence  $f_J(j', j_{\min}) \neq 0$ ). From Eq. (85), we see that we require  $j_{\max} = j + M_2 - M_1$  and  $j_{\min} = j + M_1 - M_2$ .

Taking  $M_1 = -J$  and  $M_2 = -J + \eta$ , where  $0 < \eta \leq j' - j + J$ , we have  $j_{\min} = j - \eta$ , and  $j_{\max} = j + \eta$ . For the maximum value of  $\eta$ , we find  $j_{\min} = 2j - (J + j') < j$  and  $j_{\max} = J + j'$ . We conclude that  $f_J(j', x) \neq 0$  for  $j_{\min} \leq x \leq j_{\max}$ . In particular, we see that if we have a triple  $J, j', j$  satisfying  $|J - j'| \leq j < J + j'$  and  $f_J(j', j) \neq 0$ , then the triple  $J, j', j_{\min}$  satisfies  $|J - j'| \leq j_{\min} < J + j'$  and  $f_J(j', j_{\min}) \neq 0$ .

Applying this rule recursively, we generate more triples of the form  $J, j', j_n$  where  $j_n \equiv 2j_{n-1} - (J + j') < j_{n-1}$ , and  $j_0 = j$ . Because  $j_n$  decreases at every iteration, it eventually takes the minimum possible value of  $|J - j'|$ . Consequently, we can conclude that  $f_J(j', x) \neq 0$  for

$$|J - j'| \leq x \leq J + j'.$$

It follows from Eq. (85) that for  $J, j'$  that are part of a distinguished triple, the state  $K_{J,M}|\psi\rangle$  assigns nonzero

amplitude to terms  $|j', m'\rangle$  with  $m' = x + M$  where  $x$  is in the given range. In particular, the smallest value of  $m'$  such that  $|j', m'\rangle$  receives nonzero amplitude is  $m'_{\min} = |J - j'| + M$ . However, this depends explicitly on  $M$ , contradicting the assumption, articulated in Eq. (83), that the normalized form of  $K_{J,M}|\psi\rangle$  is independent of  $M$ . QED.

#### D. Restricting to a fixed quantization axis

We restrict ourselves to the Hilbert space  $\mathcal{H}_{\hat{n}}$  containing all linear combinations of SU(2)-coherent states associated with the quantization axis  $\hat{n}$ . Given Thm. 18, an irreducible SU(2)-invariant operation on  $\mathcal{H}_{\hat{n}}$  must have a single Kraus operator  $K_J$  that can be factored as

$$K_J = S_{-J} \tilde{K}_J \quad (94)$$

where

$$\tilde{K}_J = \sum_j c_j^{(J)} |j, j\rangle \langle j, j|$$

with  $|c_j^{(J)}| \leq 1$  for all  $j$  and

$$S_{-J} = \sum_{j \geq J} |j - J, j - J\rangle \langle j, j|$$

for some positive integer or half-integer  $J$ . Clearly,  $\tilde{K}_J$  changes the relative amplitudes (weights and phases) of the  $|j, j\rangle$ , possibly eliminating the amplitude for some, while  $S_{-J}$  shifts  $|j, j\rangle$  down to  $|j - J, j - J\rangle$ .

We have now specified the form, within a given space  $\mathcal{H}_{\hat{n}}$ , of irreducible SU(2)-invariant operations that implement pure-to-pure transformations. The most general SU(2)-invariant operation that implements pure-to-pure transformations is a sum of these. Denoting the multiplicity index by  $\alpha$ , a general SU(2)-invariant operation can be written as  $\mathcal{E} = \sum_{J,\alpha} \mathcal{E}_{J,\alpha}$  where  $\mathcal{E}_{J,\alpha}$  is an irreducible SU(2)-invariant operation associated with the  $J$ th irrep. Incorporating the constraint that the trace be nonincreasing, we summarize our result with the following lemma.

**Lemma 19.** *An SU(2)-invariant operation on  $\mathcal{H}_{\hat{n}}$  that takes pure states to pure states admits a Kraus decomposition  $\{K_{J,\alpha}\}$ , of the form*

$$K_{J,\alpha} = S_{-J} \tilde{K}_{J,\alpha}$$

where  $\tilde{K}_{J,\alpha} = \sum_j c_j^{(J,\alpha)} |j, j\rangle \langle j, j|$  changes the relative amplitudes of the  $|j, j\rangle$  states, possibly eliminating some, and  $S_{-J} = \sum_{j \geq J} |j - J, j - J\rangle \langle j, j|$  shifts the value of  $j$  downward by  $J$ . The coefficients satisfy  $\sum_{J \leq j} \sum_{\alpha} |c_j^{(J,\alpha)}|^2 \leq 1$  for all  $j$ , with equality if the operation is trace-preserving.

Notice that the form of the allowed Kraus operators for an SU(2)-invariant operation on  $\mathcal{H}_{\hat{n}}$  is almost the same

as the allowed Kraus operators for an irreducible  $U(1)$ -invariant operation, discussed in Sec. III, with  $j$  playing the role of  $n$ . The only difference is that the  $j$  value can only be shifted downward, whereas  $n$  can be shifted in either direction. The resource theory of  $SU(2)$  frames in  $\mathcal{H}_{\hat{n}}$  is consequently very close to that of  $U(1)$  frames, particularly for single-copy transformations. We therefore lean heavily on the results and proofs provided in Sec. III in describing and justifying the  $SU(2)$  resource theory.

Although the Stinespring dilation theorem guarantees that there is a way of physically implementing these  $SU(2)$ -invariant operations by introducing ancillae in  $SU(2)$ -invariant states, implementing  $SU(2)$ -invariant unitaries, and tracing out systems, it is instructive to see how the shift operation is achieved in this way. In order to shift the  $j$  value down by  $J > 0$  (i.e. to implement the operation  $S_{-J}(\cdot)S_{-J}^\dagger$ ), one simply adds an ancilla in state  $|0, 0\rangle$  (the only pure  $SU(2)$ -invariant state), implements the unitary  $|j, j\rangle|0, 0\rangle \rightarrow |j - J, j - J\rangle|J, J\rangle$  (which is an  $SU(2)$ -invariant operation), and discards the ancilla.

At first sight, one might hope to shift the  $j$  value upward by the reverse of this process. However, one would be required to begin by adding an ancilla in the state  $|J, J\rangle$  where  $J > 0$ , and this operation cannot be accomplished under the  $SU(2)$ -SSR because  $|J, J\rangle$ , unlike  $|0, 0\rangle$ , does not come for free. The difference between the  $SU(2)$  and  $U(1)$  cases – shifts being permitted in both directions for the former and only downward for the latter – is a result of the fact that every number eigenstate  $|n\rangle$  can be prepared under the SSR, whereas among the  $|j, j\rangle$  states only the singlet can be prepared under the SSR.

### E. Deterministic single-copy transformations

We assume the states to be in the standard forms  $|\psi\rangle = \sum_j \sqrt{p_j}|j, j\rangle$  and  $|\phi\rangle = \sum_j \sqrt{q_j}|j, j\rangle$ .

**Theorem 20.** *The necessary and sufficient condition for the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  to be possible by a deterministic  $SU(2)$ -invariant operation is that*

$$p_j = \sum_J w_J q_{j+J}, \quad (95)$$

where the sum varies over all positive integers and half-integers and the  $w_J$  form a probability distribution.

The proof is simply the one presented in Sec. III for the associated  $U(1)$  result but where  $k$  is substituted with  $J$ .

### F. Stochastic single-copy transformations

Again, the situation is analogous to that of the  $U(1)$  case and consequently it is useful to define the  $j$ -spectrum of  $|\psi\rangle$  as the set of  $j$  values to which  $|\psi\rangle$  assigns nonzero probability. If  $|\psi\rangle = \sum_j \sqrt{p_j}|j, j\rangle$ , then the set is

$\{j|p_j \neq 0\}$ . The cardinality of this set will again be denoted by  $\mathcal{S}(\psi)$ , and a list of the elements of the set in ascending order will be denoted

$$j\text{-Spec}(\psi) \equiv (j_1(\psi), j_2(\psi), \dots, j_{\mathcal{S}(\psi)}(\psi)).$$

The conditions under which a stochastic single-copy transformation is possible are as follows.

**Theorem 21.** *The transformation  $|\psi\rangle \rightarrow |\phi\rangle$  is possible using stochastic  $SU(2)$ -invariant operations if and only if*

$$\exists J \in \{0, 1/2, 1, \dots\} : j\text{-Spec}(\phi) \subset j\text{-Spec}(\psi) - J. \quad (96)$$

Again, the proof follows the one described in the  $U(1)$  case, but where the shifts in  $j$ , unlike the shifts in  $n$ , can only be made in the downward direction.

Finally, we also have a result concerning the maximum probability of transformation which parallels Thm. 5.

**Theorem 22.** *If there is only a single value of  $J$  such that the condition  $j\text{-Spec}(\phi) \subset j\text{-Spec}(\psi) - J$  holds, then the maximum probability of achieving the transformation  $|\psi\rangle \rightarrow |\phi\rangle$  using  $SU(2)$ -invariant operations is*

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \min_j \left( \frac{p_j}{q_{j-J}} \right).$$

### G. Stochastic $SU(2)$ -frameness monotones

Every stochastic  $U(1)$ -frameness monotone that was identified in Sec. III E, has an analogue in the case of the  $SU(2)$ -SSR for states restricted to  $\mathcal{H}_{\hat{n}}$ . We need only identify  $|j, j\rangle$  with  $|n\rangle$  to define them. However, in addition to these, there are novel stochastic frameness monotones stemming from the fact that the  $j$ -spectrum can only be shifted downward. For instance, the highest  $j$  value in the spectrum,  $j_{\mathcal{S}(\psi)}(\psi)$  clearly cannot be increased and consequently is a stochastic  $SU(2)$ -frameness monotone.

### H. Asymptotic transformations

We now discuss the asymptotic limit, where we are interested in transformations of the form

$$|\psi\rangle^{\otimes n} \rightarrow |\varphi\rangle^{\otimes m} \quad (97)$$

in the limit where  $n$  and  $m$  go to infinity. (We switch from the uppercase  $N$  and  $M$  of Thms. 7 and 15 to lowercase  $n$  and  $m$  in order to avoid confusion with the azimuthal angular momentum quantum number  $M$ .)

Similarly to the  $U(1)$  case, we will see that if  $|\psi\rangle$  has a gapless  $j$ -spectrum, then  $|\psi\rangle^{\otimes n}$  has weights on  $j$  that are Gaussian in the limit  $n \rightarrow \infty$ , and given that the mean and variance of Gaussian states are additive under tensor product, it follows that the only features of  $|\psi\rangle$  and  $|\varphi\rangle$  that will be significant are the mean and variance of the

distribution over  $j$  that they define. We therefore begin by providing precise definitions of these quantities and demonstrating that they are ensemble  $SU(2)$ -frameness monotones.

Let  $|\psi\rangle$  be a state in  $\mathcal{H}_{\hat{n}}$ . By analogy to the number operator  $\hat{N}$  in the  $U(1)$  setting, we define an operator  $\mathcal{J}$  on  $\mathcal{H}_{\hat{n}}$  as

$$\mathcal{J} \equiv \sum_{j=0, \frac{1}{2}, 1, \dots} j |j, j\rangle \langle j, j|.$$

**Definition:** The mean of  $\mathcal{J}$  for the state  $|\psi\rangle$  is:

$$\mathcal{M}(|\psi\rangle) \equiv 2 \langle \psi | \mathcal{J} | \psi \rangle.$$

**Definition:** The variance in  $\mathcal{J}$  for the state  $|\psi\rangle$  is:

$$V(|\psi\rangle) \equiv 4 [\langle \psi | \mathcal{J}^2 | \psi \rangle - \langle \psi | \mathcal{J} | \psi \rangle^2].$$

The factors in the definitions of  $\mathcal{M}$  and  $V$  have been chosen such that  $\mathcal{M}(|+\rangle) = 1$  and  $V(|+\rangle) = 1$ , where  $|+\rangle \equiv (|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$ .

Lemma 8 of Sec. III F proves that the variance in  $\hat{N}$  is an ensemble monotone under the  $U(1)$ -SSR. A comparison of Eqs. (32) and (94) shows that by identifying  $|j, j\rangle$  with  $|n\rangle$ , the  $SU(2)$ -invariant pure-to-pure transformations on the space  $\mathcal{H}_{\hat{n}}$  are mathematically a proper subset of the  $U(1)$ -invariant pure-to-pure transformations, given that they do not allow upward shifts of  $j$ . It follows that the variance in  $\mathcal{J}$  is an ensemble frameness monotone over the pure states of  $\mathcal{H}_{\hat{n}}$  under the  $SU(2)$ -SSR.

Such a result also holds for the mean of  $\mathcal{J}$ .

**Lemma 23.**  $\mathcal{M}(|\psi\rangle)$  is an ensemble frameness monotone on  $\mathcal{H}_{\hat{n}}$ .

**Proof.** An  $SU(2)$ -invariant measurement transforms a state  $|\psi\rangle$  into an ensemble of states. For the most general such measurement, each outcome may be associated with an  $SU(2)$ -invariant operation that has multiple Kraus operators. However, as argued in the proof of lemma 8, it suffices to consider the measurements for which each outcome is associated with a single Kraus operator.

Suppose the outcome  $\mu$  occurs with probability  $w_\mu$  and is associated with a Kraus operator  $K_\mu$  which, by lemma 19, has the form

$$K_\mu = \sum_j c_j^{(\mu)} |j - J_\mu, j - J_\mu\rangle \langle j, j|,$$

where  $c_j^{(\mu)}$  are complex coefficients and  $J_\mu$  is a non-negative integer or half-integer. Note that

$$[\mathcal{J}, K_\mu] = -J_\mu K_\mu.$$

After an outcome  $\mu$  has occurred, the state of the system is  $|\phi_\mu\rangle = \frac{1}{\sqrt{w_\mu}} K_\mu |\psi\rangle$ , where  $w_\mu = \|K_\mu |\psi\rangle\|^2$ . It follows

that the average of the mean of  $\mathcal{J}$  is

$$\begin{aligned} \sum_\mu w_\mu \mathcal{M}(|\phi_\mu\rangle) &= \sum_\mu \langle \psi | K_\mu^\dagger \mathcal{J} K_\mu | \psi \rangle \\ &= \sum_\mu \langle \psi | K_\mu^\dagger K_\mu (\mathcal{J} - J_\mu) | \psi \rangle \\ &\leq \langle \psi | \left( \sum_\mu K_\mu^\dagger K_\mu \right) \mathcal{J} | \psi \rangle \leq \mathcal{M}(|\psi\rangle), \end{aligned} \quad (98)$$

where we have used the fact that  $\sum_\mu K_\mu^\dagger K_\mu \leq I$ . Therefore, the mean of  $\mathcal{J}$  is non-increasing on average under  $SU(2)$ -invariant operations. QED.

Another important property of the mean and the variance of  $\mathcal{J}$  is that they are both strongly additive. That is, one can easily check that for any two states  $|\psi\rangle, |\varphi\rangle \in \mathcal{H}_{\hat{n}}$  we have

$$\begin{aligned} \mathcal{M}(|\psi\rangle \otimes |\varphi\rangle) &= \mathcal{M}(|\psi\rangle) + \mathcal{M}(|\varphi\rangle) \\ V(|\psi\rangle \otimes |\varphi\rangle) &= V(|\psi\rangle) + V(|\varphi\rangle). \end{aligned}$$

This property plays an important role in the following theorem.

**Theorem 24.** For states  $|\psi\rangle$  and  $|\varphi\rangle$  with gapless  $j$ -spectra, the transformation  $|\psi\rangle^{\otimes n} \rightarrow |\varphi\rangle^{\otimes m}$  is achievable by  $SU(2)$ -invariant operations in the limit  $n \rightarrow \infty$  with an optimal rate of

$$\lim_{n \rightarrow \infty} \frac{m}{n} = \min \left\{ \frac{\mathcal{M}(|\psi\rangle)}{\mathcal{M}(|\varphi\rangle)}, \frac{V(|\psi\rangle)}{V(|\varphi\rangle)} \right\}. \quad (99)$$

The proof is provided at the end of the section.

In some resource theories, the rate of interconversion between any two states is provided by a single function over those states. For instance, in pure state entanglement theory, the entropy of entanglement is such a function, while in the resource theory of pure states under the  $Z_2$ -SSR,  $F^\infty$  of Eq. (64) is such a function, and in the resource theory of pure states with gapless number spectra under the  $U(1)$ -SSR, the variance is such a function. What the theorem above shows is that in the resource theory of pure states with gapless  $j$ -spectra under the  $SU(2)$ -SSR, no such function can be found. Rather, one requires a pair of distinct functions over the pure states, namely, the mean of  $\mathcal{J}$  and the variance of  $\mathcal{J}$ , in order to deduce the rate of interconversion between any two states.

Note that if  $\mathcal{M}(|\psi\rangle)/\mathcal{M}(|\varphi\rangle) \neq V(|\psi\rangle)/V(|\varphi\rangle)$ , then the asymptotic rate of interconversion from  $|\psi\rangle$  to  $|\varphi\rangle$  will not be the inverse of the rate from  $|\varphi\rangle$  to  $|\psi\rangle$ . To see this, simply note that the rate in one direction is  $R(\psi \rightarrow \varphi) = \min \left( \frac{\mathcal{M}(|\psi\rangle)}{\mathcal{M}(|\varphi\rangle)}, \frac{V(|\psi\rangle)}{V(|\varphi\rangle)} \right)$  while in the other direction it is  $R(\varphi \rightarrow \psi) = \min \left( \frac{\mathcal{M}(|\varphi\rangle)}{\mathcal{M}(|\psi\rangle)}, \frac{V(|\varphi\rangle)}{V(|\psi\rangle)} \right) = \left[ \max \left( \frac{\mathcal{M}(|\psi\rangle)}{\mathcal{M}(|\varphi\rangle)}, \frac{V(|\psi\rangle)}{V(|\varphi\rangle)} \right) \right]^{-1}$ . But if the asymptotic rates are not inverses of one another, then the transformation is not reversible.



Conversely, if the ratio of means is equal to the ratio of variances, then the rates are indeed inverses of one another. This result is summarized by the following corollary.

**Corollary 25.** *For states  $|\psi\rangle$  and  $|\varphi\rangle$  with gapless  $j$ -spectra, the transformation  $|\psi\rangle^{\otimes n} \rightarrow |\varphi\rangle^{\otimes m}$  can be achieved reversibly if and only if*

$$\frac{\mathcal{M}(|\psi\rangle)}{V(|\psi\rangle)} = \frac{\mathcal{M}(|\varphi\rangle)}{V(|\varphi\rangle)}.$$

*In that case  $\lim_{n \rightarrow \infty} \frac{m}{n} \equiv \mathcal{M}(|\psi\rangle)/\mathcal{M}(|\varphi\rangle) = V(|\psi\rangle)/V(|\varphi\rangle)$  is the asymptotic rate of interconversion.*

We can therefore separate the set of all pure states with gapless  $j$ -spectra into equivalence classes where the equivalence relation is equality of the ratio of  $\mathcal{M}$  to  $V$ . Within each equivalence class, reversible asymptotic interconversions are possible and either  $\mathcal{M}$  or  $V$  can serve as the unique measure of framedness (from which the asymptotic rate of interconversion between any two states can be inferred). Asymptotic interconversion of states in distinct equivalence classes can only be achieved irreversibly.

The question arises of whether one can find, in each equivalence class, a natural convention for a “gold standard” against which states can be compared. One possibility is the state

$$\left| +_p^{(j)} \right\rangle \equiv \sqrt{p} |0, 0\rangle + \sqrt{1-p} |j, j\rangle,$$

for which  $\mathcal{M}(|+_p^{(j)}\rangle) = 2(1-p)j$  and  $V(|+_p^{(j)}\rangle) = 4p(1-p)j^2$ , so that the ratio

$$\mathcal{M}(|+_p^{(j)}\rangle)/V(|+_p^{(j)}\rangle) = \frac{1}{2pj}.$$

Consequently, if we choose  $j = \lceil 1/r \rceil$ , where  $\lceil x \rceil$  denotes the smallest integer larger than  $x$ , and if we choose  $p$  such that  $p = 1/2rj$ , so that  $p \leq 1/2$  by definition, then the state  $|+_p^{(j)}\rangle$  can be taken as the gold standard for the equivalence class with ratio  $r$ .

Note that for  $r \geq 1$ , we have  $j = 1$ , the standard state is of the form  $(1/\sqrt{2r}) |0, 0\rangle + \sqrt{(2r-1)/2r} |1, 1\rangle$  and the rate at which a state  $|\psi\rangle$  can be converted to this standard is simply  $\mathcal{M}(|\psi\rangle)(r/(2r-1)) = V(|\psi\rangle)(r^2/(2r-1))$ . This convention is particularly nice at  $r = 1$ , where the standard state is  $(|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$  and the rate is simply  $\mathcal{M}(|\psi\rangle) = V(|\psi\rangle)$ .

**Proof of theorem 24.** Because both  $\mathcal{M}$  and  $V$  are ensemble framedness monotones, if the transformation is achievable then we must have  $\mathcal{M}(|\psi\rangle^{\otimes n}) \geq \mathcal{M}(|\phi\rangle^{\otimes m})$  and  $V(|\psi\rangle^{\otimes n}) \geq V(|\phi\rangle^{\otimes m})$  which is equivalent to condition (99). This establishes the necessity of the condition. We now demonstrate its sufficiency.

Following reasoning parallel to that presented in the proof of Thm. 7 (but refraining from shifting the  $j$  value

just yet), we can take  $|\psi\rangle^{\otimes n}$  to the standard form

$$|\psi\rangle^{\otimes n} = \sum_{j=n_{j_{\text{low}}}}^{n_{j_{\text{high}}}} \sqrt{r_j} |j, j\rangle, \quad (100)$$

where  $j_{\text{low}} \equiv j_1(\psi)$  and  $j_{\text{high}} \equiv j_{\mathcal{S}(\psi)}(\psi)$  and

$$r_j \equiv \sum \frac{j!}{n_{j_{\text{low}}}! n_{j_{\text{low}}+1}! \cdots n_{j_{\text{high}}}!} p_{j_{\text{low}}}^{n_{j_{\text{low}}}} p_{j_{\text{low}}+1}^{n_{j_{\text{low}}+1}} \cdots p_{j_{\text{high}}}^{n_{j_{\text{high}}}}, \quad (101)$$

where the sum is taken over all sets of nonnegative integers  $n_{j_{\text{low}}}, n_{j_{\text{low}}+1}, \dots, n_{j_{\text{high}}}$  for which  $\sum_{j'=j_{\text{low}}}^{j_{\text{high}}} n_{j'} = n$  and  $\sum_{j'=j_{\text{low}}}^{j_{\text{high}}} j' n_{j'} = j$ . In the limit  $n \rightarrow \infty$ , the  $r_j$  approach a Gaussian distribution as long as for all  $j \in \{j_{\text{low}}, \dots, j_{\text{high}}\}$ ,  $p_j > 0$  [10]. The proof is blocked if  $p_j = 0$  for some  $j$  in this range and it is for this reason that our theorem is restricted to pure states with gapless  $j$ -spectra.

First, suppose  $V(|\varphi\rangle)/V(|\psi\rangle) \geq \mathcal{M}(|\varphi\rangle)/\mathcal{M}(|\psi\rangle)$ , so that  $n/m = V(|\varphi\rangle)/V(|\psi\rangle)$ . In this case, the  $m$ -fold product of  $|\varphi\rangle$  has the same variance as the  $n$ -fold product of  $|\psi\rangle$ , but a smaller mean value of  $j$ . However, by lemma 19 one can always reduce the mean value of  $\mathcal{J}$  by any integer or half-integer amount, and this operation does not affect the variance. Implementing such a shift leaves one with a state that is arbitrarily close to  $m$  copies of  $|\varphi\rangle$  in the limit of  $n \rightarrow \infty$ . To see that this is the case, define

$$J_0 \equiv \lfloor n\mathcal{M}(|\psi\rangle) - m\mathcal{M}(|\varphi\rangle) \rfloor$$

where  $\lfloor x \rfloor$  denotes the largest integer or half-integer less than  $x$ , and define

$$|\gamma_n\rangle \equiv S_{-J_0} (|\psi\rangle^{\otimes n}).$$

Clearly,

$$\begin{aligned} \mathcal{M}(|\gamma_n\rangle) &= n\mathcal{M}(|\psi\rangle) - J_0 \\ &\rightarrow m\mathcal{M}(|\varphi\rangle) \end{aligned}$$

in the limit of  $n \rightarrow \infty$ .

The alternative is that  $V(|\varphi\rangle)/V(|\psi\rangle) \leq \mathcal{M}(|\varphi\rangle)/\mathcal{M}(|\psi\rangle)$ , so that  $n/m = \mathcal{M}(|\varphi\rangle)/\mathcal{M}(|\psi\rangle)$ . In this case, the  $m$ -fold product of  $|\varphi\rangle$  has the same mean value of  $\mathcal{J}$  as the  $n$ -fold product of  $|\psi\rangle$  but a smaller variance. All that remains to show therefore is that, using  $\text{SU}(2)$ -invariant operations, one can reduce the variance by an arbitrary amount while preserving the mean value of  $\mathcal{J}$ . (Note that such an operation does not lead to an increase of either  $V$  or  $\mathcal{M}$  and so is consistent with the latter being ensemble framedness monotones.)

The requisite operation involves implementing a measurement on each copy of  $|\psi\rangle$ . Suppose that the outcomes of the measurement are labeled by  $\mu$ , the probability of outcome  $\mu$  is denoted by  $w_\mu$  and the normalized final

state associated with outcome  $\mu$  is denoted by  $|\psi_\mu\rangle$ . We begin by showing that there exists a measurement on  $|\psi\rangle$  such that the ensemble of final states has, on average, mean of  $\mathcal{J}$  equal to that of  $|\psi\rangle$ ,

$$\sum_{\mu} w_{\mu} \mathcal{M}(|\psi_{\mu}\rangle) = \mathcal{M}(|\psi\rangle), \quad (102)$$

and a variance satisfying

$$\sum_{\mu} w_{\mu} V(|\psi_{\mu}\rangle) = \frac{m}{n} V(|\varphi\rangle). \quad (103)$$

We assume that each outcome  $\mu$  is associated with an operation  $\mathcal{E}_{\mu}$  defined by a single Kraus operator of the form  $K_{\mu} = \sum_j c_j^{(\mu)} |j\rangle \langle j|$ . This is an  $SU(2)$ -invariant measurement, by virtue of the Kraus operators being of the form outlined in lemma 19. However, for no outcome  $\mu$  does the measurement incorporate a nontrivial shift operation  $S_{-j}$ . The constraint that the overall operation be trace-preserving implies that  $\sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = I$ , or equivalently, that  $\sum_{\mu} |c_j^{(\mu)}|^2 = 1$  for all  $j$ . Note that this constraint is satisfied if one takes  $c_j^{(\mu)} = u_{j\mu}$  where  $u$  is a unitary matrix. Such a measurement does not change the mean of  $\mathcal{J}$  on average because it saturates the inequality in Eq. (98).

Recalling that the maximum  $j$  value to which  $|\psi\rangle$  assigns nonzero probability is  $j_{\max}$ , it suffices to consider the operation  $\mathcal{E}_{\mu}$  on  $\text{span}(|j\rangle \langle j| : j \leq j_{\max})$ . If the  $c_j^{(\mu)}$  are to be the components of a unitary matrix, then the range of  $\mu$  must also be 0 to  $j_{\max}$ . Now consider two extreme cases.

(i)  $\mathcal{E}_{\mu}$  is defined by the unitary matrix  $u_{j\mu} = \delta_{j,\mu}$ . In this case,  $|\psi_{\mu}\rangle = |j = \mu\rangle \langle j = \mu|$  so that  $V(|\psi_{\mu}\rangle) = 0$  for all  $\mu$ , and consequently  $\sum_{\mu} w_{\mu} V(|\psi_{\mu}\rangle) = 0$ . Meanwhile, the mean of  $\mathcal{J}$  is the same on average,  $\sum_{\mu} w_{\mu} \mathcal{M}(|\psi_{\mu}\rangle) = \sum_{\mu} w_{\mu} \mu = \sum_{\mu} |\langle \mu | \psi \rangle|^2 \mu = \mathcal{M}(|\psi\rangle)$ .

(ii)  $\mathcal{E}_{\mu}$  is defined by the unitary Fourier matrix  $u_{j\mu} = j_{\max}^{-1/2} \exp[i2\pi\mu j/j_{\max}]$ . In this case,  $|\psi_{\mu}\rangle$  differs from  $|\psi\rangle$  only by the phases of the  $|j\rangle$  terms, so that  $V(|\psi_{\mu}\rangle) = V(|\psi\rangle)$  for all  $\mu$ , and consequently  $\sum_{\mu} w_{\mu} V(|\psi_{\mu}\rangle) = V(|\psi\rangle)$ .

Because there exists a continuous path between any two unitaries<sup>7</sup>, and because the average variance is a continuous function of the unitary matrix  $u$ , we conclude that for every variance in the range 0 to  $V(|\psi\rangle)$ , there exists a unitary matrix  $u$  on the path connecting  $\delta_{j,\mu}$  and  $j_{\max}^{-1/2} \exp[i2\pi\mu j/j_{\max}]$  that achieves this variance on average. In particular, we can find a measurement that yields  $\sum_{\mu} w_{\mu} V(|\psi_{\mu}\rangle) = \frac{m}{n} V(|\varphi\rangle)$ .

After performing this measurement on each of the  $n$  copies of  $|\psi\rangle$ , one obtains the final state

$$|\chi_n\rangle = |\psi_0\rangle^{\otimes n_0} \otimes |\psi_1\rangle^{\otimes n_1} \otimes \cdots \otimes |\psi_{N-1}\rangle^{\otimes n_{j_{\max}}} ,$$

where in the limit  $n \rightarrow \infty$  we have  $n_{\mu} \rightarrow w_{\mu} n$  for  $\mu = 0, 1, \dots, j_{\max}$ . Hence, in the limit  $n \rightarrow \infty$  we have

$$\begin{aligned} V(|\chi_n\rangle) &= \sum_{\mu=0}^{j_{\max}} n_{\mu} V(|\psi_{\mu}\rangle) \\ &\rightarrow n \sum_{\mu=0}^{j_{\max}} w_{\mu} V(|\psi_{\mu}\rangle) \\ &= m V(|\varphi\rangle). \end{aligned}$$

In addition,

$$\begin{aligned} \mathcal{M}(|\chi_n\rangle) &= \sum_{\mu=0}^{j_{\max}} n_{\mu} \mathcal{M}(|\psi_{\mu}\rangle) \\ &\rightarrow n \sum_{\mu=0}^{j_{\max}} w_{\mu} \mathcal{M}(|\psi_{\mu}\rangle) \\ &= n \mathcal{M}(|\psi\rangle) \\ &= \mathcal{M}(|\varphi\rangle^{\otimes m}), \end{aligned}$$

where we have used the fact that the mean of  $\mathcal{J}$  is unchanged as  $|\psi\rangle \rightarrow |\psi_{\mu}\rangle$ . Hence, in the limit  $n \rightarrow \infty$ ,  $|\chi_n\rangle$  and  $|\varphi\rangle^{\otimes m}$  have the same mean value of  $\mathcal{J}$  and the same variance.

Therefore, if it can be shown that both  $|\chi_n\rangle$  and  $|\varphi\rangle^{\otimes m}$  approach Gaussian states in the limit  $n \rightarrow \infty$ , then it follows that these approach the *same* state in this limit. Clearly,  $|\varphi\rangle^{\otimes m}$  approaches a Gaussian by the same argument establishing that  $|\psi\rangle^{\otimes n}$  does. Similarly, each factor state of  $|\chi_n\rangle$  of the form  $|\psi_{\mu}\rangle^{\otimes n_{\mu}}$  approaches a Gaussian because for each  $\mu$ ,  $n_{\mu} \rightarrow \infty$  as  $n \rightarrow \infty$ . It remains only to show that a tensor product of Gaussians is also Gaussian.

Consider the tensor product  $|\psi_1\rangle^{\otimes n_1} \otimes |\psi_2\rangle^{\otimes n_2}$ , where  $|\psi_{\mu}\rangle^{\otimes n_{\mu}} = \sum_j r_j^{(\mu)} |j, j\rangle \langle j, j|$  and the  $r_j^{(\mu)}$  are Gaussian distributions over  $j$ . Clearly,

$$|\psi_1\rangle^{\otimes n_1} \otimes |\psi_2\rangle^{\otimes n_2} = \sum_{j, j'} r_j^{(1)} r_{j'}^{(2)} |j + j', j + j'\rangle \langle j + j', j + j'|.$$

Defining  $j'' \equiv j + j'$  and  $x \equiv j - j'$ , we have

$$|\psi_1\rangle^{\otimes n_1} \otimes |\psi_2\rangle^{\otimes n_2} = \sum_{j''} \tilde{r}_{j''} |j'', j''\rangle \langle j'', j''|,$$

where

$$\tilde{r}_{j''} = \sum_x r_{(j''+x)/2}^{(1)} r_{(j''-x)/2}^{(2)}.$$

In the limit of  $n \rightarrow \infty$ , this is a convolution of two Gaussians, which is also a Gaussian. Note that the variance

<sup>7</sup> We thank Larry Bates, Peter Lancaster and Peter Zvengrowski for bringing this to our attention; in particular we thank Larry Bates and Peter Lancaster for showing us (explicitly) several different continuous paths that connect the Fourier matrix  $u_{j\mu} = j_{\max}^{-1/2} \exp[i2\pi\mu j/j_{\max}]$  with the identity.

(respectively mean) of the convolution is equal to the sum of the variances (respectively means) of the components, as is required for consistency with the additivity of  $V$  and  $\mathcal{M}$  under tensor product. The argument clearly generalizes to the tensor product of an arbitrary number of Gaussians, implying that  $|\chi_n\rangle$  approaches a Gaussian. QED.

## VI. CONCLUSIONS

A superselection rule is a restriction on operations. It may arise from the practical circumstance of lacking a reference frame for some degree of freedom. The nature of this degree of freedom – in particular its associated symmetry group – determines the set of operations that are forbidden by the superselection rule. Superselection rules therefore admit of degree: the more operations they forbid, the stronger they are.

There is a strict ordering by strength of the three SSRs we consider in this article. If  $\mathfrak{D}[G]$  denotes the set of operations that are forbidden under a  $G$ -SSR, then  $\mathfrak{D}[Z_2] \subset \mathfrak{D}[U(1)] \subset \mathfrak{D}[SU(2)]$ . At an abstract level, this clearly follows from the fact that  $Z_2$  is a subgroup of  $U(1)$  which is a subgroup of  $SU(2)$ . A physical explanation of the ordering, however, requires us to go beyond the particular restrictions – lack of reference frames for chirality, optical phase, and orientation – that we have chosen to emphasize in this article as illustrations of each type of SSR. (For instance, the operations that are forbidden by lacking a frame for chirality are not a subset of those forbidden by lacking a frame for orientation because without a shared reference frame for chirality, Bob cannot tell whether a glove he receives from Alice would be described as left or right by her, whereas if he lacks a shared reference frame for orientation, he can still do so.)

Fortunately, a triple of restrictions that *do* provide a physical explanation of the ordering can easily be provided. As noted in the article, in addition to its significance in optics, the  $U(1)$ -SSR also describes the restriction that Alice and Bob face when they share a single direction in space. Taking  $\hat{z}$  to be their shared axis, what they lack is knowledge of the angle between their  $\hat{x}$  axes. Operations that are forbidden when sharing a single axis are a strict subset of those that are forbidden when sharing no axis (the restriction leading to an  $SU(2)$ -SSR). Similarly, in addition to characterizing the lack of a chiral reference frame, a  $Z_2$ -SSR characterizes the restriction that arises if Alice and Bob share a  $\hat{z}$ -axis and know the angle modulo  $\pi$  between their respective  $\hat{x}$ -axes. In this case, they certainly know more than if they knew nothing of the angle between their  $\hat{x}$ -axes and consequently the operations that are forbidden are a strict subset of those that arise in the latter case. In summary, an  $SU(2)$ -SSR is a stronger restriction than that of a  $U(1)$ -SSR which in turn is stronger than that of a  $Z_2$ -SSR.

We have shown that the extent of manipulations that

one can make upon the resources defined by each restriction (quantum states that stand in for the missing reference frames) depends on the strength of the restriction. Given a single copy of any pure state that acts as a  $Z_2$ -resource, there is a nonzero probability of transforming it into a single copy of any other such state. By contrast, for pure states that act as  $U(1)$ -resources, there are many pairs for which such a transformation is not possible (in either one or both directions). The impossible cases are even more numerous for pure states that act as  $SU(2)$ -resources. Similarly, arbitrarily many copies of any pure state that acts as a  $Z_\infty$ -resource can be transformed *reversibly* into any other such state with some nonzero rate, whereas only for certain classes of pure  $U(1)$ -resource states is such asymptotic reversible interconversion possible, and for pure  $SU(2)$ -resource states, the classes are smaller still.

The resource theory for quantum reference frames therefore provides another example, in addition to that of the resource theory of multipartite entanglement, of the generic phenomenon that the ease of resource manipulations decreases with the strength of the restriction.

There are a great many open questions that remain concerning the manipulation of quantum reference frames. In the context of phase references, the problem of finding the maximum probability with which one can transform a single copy of some state into a single copy of another has only been solved under restrictive conditions. Furthermore, the problem of characterizing when asymptotic transformations can be achieved with nonzero rate and when they can be achieved reversibly has only been solved completely for states with gapless number spectra. Similar comments apply for the subset  $\mathcal{C}_n$  of Cartesian frame states.

Extending our results to arbitrary states in the  $SU(2)$ -resource theory is likely to be a very difficult task. Note, however, that a feature of the subspace  $\mathcal{H}_{\hat{n}}$  that simplifies the resource analysis is that it is closed under tensor products. Another subspace that is closed in this fashion is  $\mathcal{H}_{\hat{n},0} \equiv \text{span}\{|j, m=0\rangle : j = 0, 1, 2, \dots\}$  because  $|j_1, m_1=0\rangle$  and  $|j_2, m_2=0\rangle$  only couple to states  $|j, m\rangle$  where  $m = m_1 + m_2 = 0$ . Furthermore, subspaces of the form  $\mathcal{H}_{\hat{n},m} \equiv \text{span}\{|j, m\rangle : j \geq m\}$  (where  $j$  values are integer or half-integer according to  $m$ ), which are simply the various eigenspaces of  $J_{\hat{n}}$ , although not closed under tensor product nonetheless have the nice feature that the tensor product of a state from  $\mathcal{H}_{\hat{n},m}$  and a state from  $\mathcal{H}_{\hat{n},m'}$  is confined to  $\mathcal{H}_{\hat{n},m+m'}$ . It follows that the theories of frame manipulations on these subspaces are likely to be more tractable than the completely general theory and consequently a promising avenue for future research.

Another direction in which this work may be extended is towards resource theories for reference frames associated with other groups. Any sort of reference frame can be considered, but particularly interesting possibilities include: reference orderings (associated with the permutation group) [38, 39, 40], inertial frames (associated with

the Lorentz group) [41], frames for global positioning (associated with the Heisenberg-Weyl group) [42], or even exotic possibilities such as frames for the color degree of freedom in quantum chromodynamics (associated with  $SU(3)$ ).

There are also many aspects of resource theories that we have not addressed here. For instance, this article has only been concerned with single-copy and asymptotic transformations. Transformations between multiple but finite numbers of copies have not been considered. More importantly, we have restricted our attention to pure states. In practice, resources are always mixed to some extent and one of the some significant problems is to determine the extent to which one can purify a resource. Furthermore, if the experience from entanglement theory is any guide, many interesting and surprising phenomena are likely to arise in the context of mixed states. One can already see, however, that the parallels to entanglement theory will be limited. Specifically, because there are no  $SU(2)$ -invariant pure states with  $j > 0$ , a mixed  $SU(2)$ -invariant state in a subspace  $\mathcal{H}_j$  with  $j > 0$  does not admit any convex decomposition into  $SU(2)$ -invariant pure states. It follows that although the  $SU(2)$ -invariant mixed state is not a resource, the elements of every convex decomposition of this state into pure states *are* resources. It is therefore a bad idea, for instance, to attempt to define a framedness monotone for such mixed states by the convex roof extension – a framedness of formation must be defined differently from the entanglement of formation.

A strong motivation for the present work is that every novel resource theory provides an interesting new perspective on its brethren. Besides the case of quantum reference frames, resources that have seen some attention of late include: purity as a resource for doing mechanical work [43, 44], nonGaussian states as resources for overcoming a restriction to Gaussian operations [45], and nonlocal boxes or super-quantum correlations as resources in the context of quantum theory [46, 47, 48]. Even if one's interest is confined to a particular resource

theory, such as the theory of entanglement, studying alternative resource theories as foils to the one of interest may well provide a faster route to progress.

That being said, it is also hoped that the present work and its like will provide a bit of an antidote to the pernicious notion that the theory of entanglement somehow provides the deepest insights into the foundations of quantum theory. Not so; the restriction with respect to which the resource of entanglement is defined – local operations and classical communication – is a *practical* rather than a foundational restriction. The universe doesn't care especially for classical channels. *We* care because it is at present much more difficult to equip distant parties with a quantum channel than it is to equip them with one that is classical, and consequently anything that can substitute for the former given the latter is of great practical value to us. The restriction of LOCC is no different in kind from that of failing to have a sample of some particular reference frame. Nor is entanglement theory particularly distinguished: it is just one of *many* resource theories and many of its features are quite generic. It is hoped that the detailed examples provided in this article will drive this point home and prompt the quantum information community to spend less time on the increasingly esoteric details of entanglement theory and more time on exploring basic questions about other physical resources. They are likely to be rewarded with unexplored country.

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